Parabolic-like mappings

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Odi et amo. Quare id faciam, fortasse requiris.
Nescio, sed fieri sentio et excrucior.

A Sonia Venuti

Abstract

In this thesis we introduce the notion of a *parabolic-like mapping*. Such an object is similar to a polynomial-like mapping, but it has a parabolic external class, *i.e.* an external map with a parabolic fixed point. In the first part of the thesis we define the notion of parabolic-like mapping and we study the dynamical properties of parabolic-like mappings. We prove a Straightening Theorem for parabolic-like mappings which states that any parabolic-like mapping of degree 2 is hybrid conjugate to a member of the family

$$Per_1(1) = \left\{ [P_A] \,|\, P_A(z) = z + \frac{1}{z} + A,\ A \in \mathbb{C} \right\},$$

a unique such member if the filled Julia set is connected. In the second part of the thesis we study analytic families of degree 2 parabolic-like mappings $(f_\lambda)_{\lambda\in\Lambda}$. We prove that the corresponding family of hybrid conjugacies induces a continuous map $\chi : \Lambda \to \mathbb{C}$, which associates to each $\lambda \in \Lambda$ the multiplier of the fixed point of the hybrid equivalent rational map P_A. We prove that, under suitable conditions, the map χ restricts to a ramified covering from the connectedness locus of $(f_\lambda)_{\lambda\in\Lambda}$ to the connectedness locus $M_1 \setminus \{1\}$.

Dansk resumé

I denne afhandling introducer vi begrebet *parabolsk-lignende afbildning*, som er en pendant til polynomiums-lignende afbildning, men med ekstern klasse havende et parabolsk fikspunkt. I den første del af afhandlingen definerer og studerer vi dynamikken af parabolsk-lignende afbildninger. Vi viser en rektifikationssætning for parabolsk-lignende afbildninger. Sætning siger, at enhver parabolsk-lignende afbildning af grad 2 er hybrid konjugeret til en rapresentant P_A for en klasse i familien $Per_1(1) = \{[P_A] \mid P_A(z) = z+1/z+A,\ A \in \mathbb{C}\}$, og klassen $[P_A]$ er unik, hvis den udfyldte Julia mængde af den parabolsk-lignende afbildning er sammenhængde. I anden del af afhandlingen studere vi analytiske familier af parabolsk-lignende afbildning af grad 2, og deres parameter rum, i det følgende kaldet Λ. Rektifikationssætningen inducer en kontinuert afbildning $\chi : \Lambda \to \mathbb{C}$, som associer til hvert $\lambda \in \Lambda$ egenværdi af fikspunktet for den afbildning, P_A som er hybridt konjugeret til den parabolsk-lignende afbildning givet ved λ. Vi viser, at under passende forhold, har afbildningen χ en restriktion som er en forgrenet overlejring af $M_1 \setminus \{1\}$.

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Contents

Chapter 1

Introduction

We consider the iteration of a function $f : \mathbb{C} \to \mathbb{C}$ a dynamical system. Let $z_0 \in \mathbb{C}$, the *orbit* for z_0 under f is the sequence

$$\{f^n(z_0) := \underbrace{(f \circ \ldots \circ f)}_{n\ times}(z_0);\ n \in \mathbb{N}\}.$$

Particular kinds of orbits are *fixed points*, for which $f(z_0) = z_0$, and *periodic points*, for which there exists p such that $f^p(z_0) = z_0$ (and p is called the *period*). The classical theory of holomorphic dynamics begins with the study of iterates of holomorphic maps in a neighborhood of a periodic point. This is actually known as *local theory*, whose main object is to find simpler models in order to understand the dynamics, at least locally.

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. Every holomorphic function on the Riemann sphere has the form $f(z) = \frac{P(z)}{Q(z)}$, where P and Q are polynomials with no common factors. We define the degree of f as the maximum of the degree of P and that of Q. By considering the number of zeros and poles of f it is easy to see that conformal maps of the sphere are the Möbius transformations, i.e. the degree 1 rational maps.

The main activity in holomorphic dynamics is the study of the orbits of the points in $\widehat{\mathbb{C}}$. More precisely, we try to classify the points in $\widehat{\mathbb{C}}$ in terms of the asymptotic behavior of their orbits. Hence, we can begin our study by posing the following natural question: what happens to the orbit of z_0 when it is perturbed slightly? If the family $(f^n)_{n\in\mathbb{N}}$ is equicontinuous in a neighborhood of z_0, the orbit does not change much by definition. If it is not, we cannot say anything. Ascoli's theorem states that a family of functions is equicontinuous if and only if it is *normal*.

Definition 1.0.1. (Normal family) A family $\mathcal{F}$ of holomorphic functions on a domain $U \subseteq \widehat{\mathbb{C}}$, U connected and open, is said to be *normal on U*

if each sequence of functions in $\mathcal{F}$ contains a subsequence which converges uniformly on every compact subset of U.

Notice that we allow the limit to be infinity.

Theorem 1.0.2. (Ascoli's theorem) *A family of analytic functions $\mathcal{F}$ is normal if and only if $\mathcal{F}$ is equicontinuous on compact sets.*

Montel gave a characterization of normality which is simple to verify: a family of holomorphic functions is normal on a domain if the image of the domain by the family omits at least three different values.

Theorem 1.0.3. (Montel's theorem) *Let $\mathcal{F}$ be a family of analytic functions on a domain $U \subseteq \widehat{\mathbb{C}}$. If there exist three different points z_1, z_2, z_3 on the Riemann sphere such that $z_i \notin f(U),\ i \in 1, 2, 3$ for all $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.*

The concept of normality in complex dynamics is usually applied to the family of iterates of a given holomorphic map f, i.e.

$$\mathcal{F} = \{f^n; n \in \mathbb{N}\}$$

and it can be used to define a partition of $\widehat{\mathbb{C}}$ which is dynamically meaningful.

Definition 1.0.4. (Fatou and Julia sets) Let f be a rational function on $\widehat{\mathbb{C}}$. The set of points $z \in \widehat{\mathbb{C}}$ such that $\mathcal{F}$ is normal in a neighborhood of z is called *the Fatou set*, and we will denote it by F_f.
Its complementary set is called *the Julia set*, and we will denote it by J_f.

The Fatou set is open by definition, therefore the Julia set is closed. Both Fatou and Julia set are totally invariant under the dynamics of f. This means that if a point belong to the Fatou set, all its preimages and its image belong to the Fatou set too, and the same is true for the Julia set.

We call *Fatou component*, and we denote it by C, any connected component of the Fatou set.

These definitions take a special form in the case of polynomials.

1.0.1 Polynomials

Let P be a polynomial. For a polynomial, the *filled Julia set* $K(P)$ is the set of points whose orbits do not tend to infinity, which is a totally invariant set, i.e.

$$K(P) = \{z \mid P^n(z) \nrightarrow \infty\}.$$

The complement is the basin of attraction of infinity

$$A_\infty(P) = \{z \mid P^n(z) \to \infty\}.$$

The Julia set is the common boundary of the filled Julia set and A_∞

$$J(P) := \partial K(P) = \partial A_\infty(P).$$

1.1 Local theory

We say that z_0 is a periodic point of period p if $f^p(z_0) = z_0$. In that case the orbit of z_0 is called a *cycle*, and has the form $\{z_0, z_1, \ldots, z_{p-1}\}$.

We define the multiplier of the cycle as

$$\lambda = (f^p)'(z_0) = f'(z_0) \cdot f'(z_1) \cdots f'(z_p)$$

Fixed points are periodic points of period 1. Observe that z_0 is a periodic point of period p if and only if z_0 is fixed for f^p. Periodic points can be classified by the value of the multiplier λ. If:

- $0 < |\lambda| < 1$ the orbit is *attracting*;
- $|\lambda| = 0$ the orbit is *superattracting*;
- $|\lambda| > 1$ the orbit is *repelling*;
- $|\lambda| = 1$ the orbit is *indifferent*:
  - if $\lambda = e^{2\pi i p/q}$, $(p, q) = 1$, $f^q \neq Id$, then we say that the orbit is *parabolic indifferent*;
  - if $\lambda = e^{2\pi i\theta}$ with θ irrational, we say that the orbit is *irrationally indifferent*.

Preperiodic points are points z_0 which are not periodic, but for which there exists $n_0 \neq 1$ such that $f^{n_0}(z_0)$ is a periodic point.

In the rest of this section we will discuss briefly the dynamics of a holomorphic map in some neighborhood of an attracting/repelling, superattracting and parabolic fixed point. We restrict our local study to neighborhoods of fixed points, instead of periodic orbits, in order to simplify the notation. The reader is referred to [M] for a more detailed treatment of local theory in holomorphic dynamics and for the proofs of the statements.

1.1.1 Conjugacies

Conjugate functions qualitatively have the same dynamics, and thus if we have a function f that is conjugate on a set $U \subseteq \widehat{\mathbb{C}}$ to a function g, we can study the dynamics of g to know that of f on U. More precisely:

Definition 1.1.1. Let $U, U', V, V' \subseteq \widehat{\mathbb{C}}$ and $f : U' \to U,\ g : V' \longrightarrow V$ be two holomorphic functions. We say that f, g are *topologically conjugate* on $U \cup U' \subseteq \widehat{\mathbb{C}}$ if there exists $\phi : U \cup U' \longrightarrow V \cup V'$ homeomorphism such that, for all $z \in \widehat{\mathbb{C}}$

$$\phi(f(z)) = g(\phi(z))$$

If moreover ϕ is quasiconformal/conformal, we say that f, g are *quasiconformally/conformally conjugate*. In particular, if ϕ is quasiconformal with $\overline{\partial}\phi = 0$ almost everywhere on K_f we say that f, g are *hybrid conjugate*.

Hence, the goal is to find a simple function conjugate to the starting one. This problem has been solved in neighborhoods of the periodic points of a function, and has different results depending on the nature of the periodic points (attracting or repelling, superattracting, indifferent). Since a holomorphic map coincides with its Taylor expansion, and if f is conjugate to g, we can study the dynamics of f to know that of g, by conjugating (if necessary) our map with a Möbius transformation, we can consider the fixed point at $z = 0$, hence our map of the form:

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + a_4 z^4 + ...$$

1.1.2 The attracting/repelling case

In the attracting and repelling case, the dynamics are conjugate to the linear part, i.e. it is a contraction or respectively an expansion about the fixed point. For a proof of the following result we refer to [M].

Theorem 1.1.2. (König's linearization theorem) *Let f be a holomorphic map with expansion $f(z) = \lambda z + a_2 z^2 + a_3 z^3 + a_4 z^4 + ...$ If the multiplier λ satisfies $|\lambda| \neq 0, 1$, then there exists a local conformal change of coordinates $\omega = \phi(z)$, with $\phi(0) = 0$, such that $\phi \circ f \circ \phi^{-1}$ is the linear map $\omega \to \lambda\omega$ for all ω in some neighborhood of the origin. Furthermore, ϕ is unique up to multiplication by a nonzero constant.*

Definition 1.1.3. If z_0 is an attracting fixed point, we define the *basin of attraction* of z_0 as

$$\mathcal{A} = \mathcal{A}(z_0) = \{z \in \widehat{\mathbb{C}} : f^n(z) \to z_0 \text{ for } n \to \infty\}.$$

The *immediate basin of attraction* $\mathcal{A}_0$ of z_0 is the connected component of the basin which contains z_0.

In the attracting case we can restate König's linearization theorem in a more global form (see [M]):

Corollary 1.1.1. *Let f be a holomorphic map with $f(z_0) = z_0$ and $f'(z_0) = \lambda$, $0 < |\lambda| < 1$, then there exists a conformal map ϕ from $\mathcal{A}$ to $\mathbb{C}$, with $\phi(z_0) = 0$, so that the diagram*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{A} \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ \mathbb{C} & \xrightarrow{\lambda \cdot} & \mathbb{C} \end{array} \tag{1.1}$$

is commutative, and so that ϕ takes a neighborhood of z_0 biholomorphically onto a neighborhood of zero. Furthermore, ϕ is unique up to multiplication by a constant.

Hence in some small neighborhood $\mathbb{D}_\epsilon$ of 0, $\mathbb{D}_\epsilon \in \mathbb{C}$, there exists a local inverse $\psi_\epsilon : \mathbb{D}_\epsilon \to \mathcal{A}_0$, which extends to some maximal open disk $\mathbb{D}_r$ about the origin. Furthermore, ψ extends homeomorphically over the boundary $\partial\mathbb{D}_r$, and the image $\psi(\partial\mathbb{D}_r) \subset \mathcal{A}_0$ *necessarily contains a singular point of f*. This implies that for a rational map f of degree $d \geq 2$, the number of attracting fixed points (more generally, the number of attracting periodic orbits) is finite, less than or equal to the number of critical points (see [M], pg 81).

1.1.3 The superattracting case

In the supettracting case, the situation is different since there is no linear part, hence our map takes the form

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + ...,$$

where $n > 1$ is the *local degree* of f.

For a proof of the following result we refer to [M].

Theorem 1.1.4. (Böttcher) *Let f be a holomorphic map with expansion $f(z) = a_n z^n + a_{n+1} z^{n+1} + ...$, where $n > 1$. Then there exists a local conformal change of coordinates $\omega = \phi(z)$, with $\phi(0) = 0$, which conjugates f to $\omega \to \omega^n$ in a neighborhood of zero. Furthermore, ϕ is unique up to multiplication by an $(n-1)$st root of unity.*

Hence in some neighborhood of the superattracting fixed point, the map f is conjugate to

$$\phi \circ f \circ \phi^{-1} : \omega \to \omega^n,$$

where $n - 1$ is the multiplicity of the critical point $z = 0$. The map ϕ is called a *Böttcher map*. As in the attracting case, the Böttcher map has a local inverse ψ_ϵ defined in some small neighborhood $\mathbb{D}_\epsilon$ of 0. In [M], pg. 91-92, is proven:

Theorem 1.1.5. *Let f be a holomorphic map with expansion $f(z) = a_n z^n + a_{n+1} z^{n+1} + ...$, where $n > 1$, ϕ be the associated Böttcher map, and ψ_ϵ be a local inverse. Then there exists a unique open disk $\mathbb{D}_r$ of maximal radius $0 < r \leq 1$ such that ψ_ϵ extends holomorphically to a map ψ from $\mathbb{D}_r$ to the immediate basin $\mathcal{A}_0$ of the superattracting fixed point. If $r = 1$, then ψ maps the unit disk biholomorphically onto $\mathcal{A}_0$, and the superattracting fixed point is the only critical point in the basin. On the other hand, if $r < 1$ then there is at least one other critical point in $\mathcal{A}_0$, lying on the boundary of $\psi(\mathbb{D}_r)$.*

1.1.4 Application to polynomial dynamics

The Böttcher Theorem has important applications to the dynamics of polynomials, since every polynomial of degree $d \geq 2$ defined on the complex plane extends to a rational map defined on the whole Riemann sphere with infinity as superattracting fixed point of multiplicity $d - 1$. Hence we have the following theorem (for a proof see [M], pg. 96)

Theorem 1.1.6. *Let f be a polynomial of degree $d \geq 2$. If the filled Julia set K_f contains all of the finite critical points of f, then both K_f and $J_f = \partial K_f$ are connected, and the complement of K_f is conformally isomorphic to the exterior of the unit disk $\overline{\mathbb{D}}$ under an isomorphism*

$$\phi : \mathbb{C} \setminus K_f \to \mathbb{C} \setminus \overline{\mathbb{D}},$$

and such that $\phi \circ f \circ \phi^{-1} : \omega \to \omega^d$. On the other hand, if at least one critical point of f belongs to $\mathbb{C} \setminus K_f$, then both K_f and J_f have uncountably many connected components.

This theorem will be particularly useful when we will study *Polynomial-like mappings*.

1.1.5 The parabolic case

The indifferent parabolic case is more complicated to state, since there exist different directions emerging from z_0, some with attracting behavior and some with repelling dynamics, which are called *petals*, where our map is conjugate to a translation. We primarily consider the case $\lambda = 1$, hence our map of the form

$$f(z) = z(1 + az^n + ...), \ \ n \geq 1, \ a \neq 0.$$

The integer $m = n + 1$ is called the *multiplicity* of the parabolic fixed point, and the integer n is called the *degeneracy/parabolic multiplicity* of the parabolic fixed point. The multiplicity is defined to be the unique integer m for which the power series expansion of $f(z) - z$ about the parabolic fixed point z_0 takes the form:

$$f(z) - z = a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + ...$$

Note that $m \geq 2$ if and only if the multiplier at z_0 is equal to one.

Definition 1.1.7. Let f be a holomorphic map of the form $f(z) = z(1 + az^n + ...)$, $n \geq 1$, $a \neq 0$. A *complex number* v is called a repulsor vector for f at the origin (see [M] pg. 104) if $nav^n = 1$, and an attraction vector if $nav^n = -1$. There are n equally spaced attraction vectors at the origin, separated by n equally spaced repulsor vectors.

Let N be some neighborhood of the origin, where our map f is defined and univalent, and let N' be its image under f. An open set $P_j \subset N$ is called an *attracting petal* for f for the direction v_j at the parabolic fixed point if $f(P_j) \subset P_j$ and an orbit of f is eventually absorbed by P_j if and only if it converges to the parabolic fixed point from the direction v_j.

On the other hand, an open set $P_k \subset N$ is called a *repelling petal* for f for the repulsor vector v_k if P_k is an attracting petal for f^{-1} for the vector v_k.

The *parabolic basin of attraction* $\mathcal{A}_j$ associated to the attraction vector v_j is the set of points for which the orbit is eventually absorbed by P_j.

If the multiplier of the parabolic fixed point is $\lambda = e^{2\pi i p/q}$, $(p, q) = 1$, then the number of attraction and repulsor vectors at the parabolic fixed point is a multiple of q, since the multiplicity $m = n+1$ of $z = 0$ as parabolic fixed point of f^q is congruent to 1 modulo q (see [M] pg. 109).

The following is the Leau-Fatou Theorem, a proof of which can be found in [M] pg. 112.

Theorem 1.1.8. *If z_0 is a parabolic fixed point of multiplicity $m = n+1 \geq 2$, then within any neighborhood of z_0 there exist simply connected petals Ξ_j,* $0 \leq$

$j \leq 2n-1$, where Ξ_j is either repelling or attracting according to whether j is even or odd. Furthermore, these petals can be chosen such that the union

$$\{z_0\} \cup \Xi_0 \cup \ldots \cup \Xi_{2n-1}$$

is a neighborhood of z_0. When $n > 1$, each Ξ_j intersects each of its two immediate neighbors in a simply connected region $\Xi_j \cap \Xi_{j\pm1}$, but it is disjoint from the remaining Ξ_k.

Hence, in a neighborhood of a parabolic fixed point of degeneracy/parabolic multiplicity n, there are n attracting petals which alternate with n repelling petals. On each petal the map f is conjugate to a translation:

Theorem 1.1.9. *For any attracting or repelling petal Ξ, there is one and, up to composition with a translation, only one conformal embedding $\phi : \Xi :\to \mathbb{C}$ which satisfies the* ***Abel functional equation***

$$\phi(f(z)) = 1 + \phi(z)$$

for all $z \in \Xi \cap f^{-1}(\Xi)$.

The map ϕ is called a *Fatou coordinate for the petal* Ξ. By an iterative local change of coordinates applied to eliminate lower order terms one by one, we obtain conformal diffeomorphisms g which conjugate f to the map $\hat{f}(z) = z(1 + z^n + cz^{2n} + O(z^{3n}))$ on Ξ. Then, the Fatou coordinates take the form:

$$\phi(z) = \Phi \circ I(z),$$

where

$$w = I(z) = -\frac{1}{naz^n}$$

conjugates the map $\hat{f}$ with the map

$$f^*(w) = w + 1 + \frac{c}{w} + O(\frac{1}{w^2}),$$

(where c is a constant); and in [Sh] is proven that

$$\Phi(z) = z - c\, log z + c' + o(1),$$

and

$$\Phi'(z) = 1 + o(1).$$

Often it is convenient to consider the quotient of a petal Ξ under the equivalence relation identifying z and $f(z)$ if both z and $f(z)$ belong to Ξ. This quotient manifold is called the *Ecalle cilinder*, and it is conformally isomorphic to the infinite cylinder $\mathbb{C}/\mathbb{Z}$ (for a proof of the following theorem see [M] pg. 113-117).

Theorem 1.1.10. *For each attracting or repelling petal Ξ, the quotient manifold Ξ/f is conformally isomorphic to the infinite cylinder $\mathbb{C}/\mathbb{Z}$.*

Finally, we state the following result, a proof of which can be found in [M] pg. 120.

Theorem 1.1.11. *If z_0 is a parabolic fixed point with multiplier $\lambda = 1$, then each immediate basin for z_0 contains at least one critical point of f. Furthermore, each basin contains one and only one attracting petal Ξ_{max} which maps univalently onto some right half-plane under ϕ and which is maximal with respect this property. This preferred petal Ξ_{max} always has one or more critical points on its boundary.*

1.2 Global theory

As we saw, critical points play an important role in complex dynamical systems because in each basin of attraction for a parabolic, attracting or superattracting periodic point there must be a critical point. Rotation domains also require a critical orbit which accumulates on their boundery. All these results are due to several authors from Fatou and Julia to Mañe, Shishikura and Epstein. In particular, Shishikura proved (see [Sh1]), as a consequence, that the number of non repelling cycles is bounded by the number of critical points of the function, and Epstein refined this inequality (see [E]).

Theorem 1.2.1. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree $d \geq 2$, then the number of attracting or indifferent cycles is at most $2d - 2$.*

Hence most periodic orbits repel. The following Proposition states to which set (Fatou or Julia) the orbits belong, depending on the nature of the orbit.

Proposition 1.2.2. *All attracting periodic orbits and their basins of attraction belong to the Fatou set.*
Let Ω be an attracting basin, then $\partial\Omega$ belongs to the Julia set.
Repelling periodic orbits belong to the Julia set.
Parabolic points belong also to the Julia set.
Irrational indifferent points may belong to the Julia or to the Fatou set.

The following result was proved in different ways by both Fatou and Julia (both proofs, adapted to our terminology, are found in [M] pg. 156-158).

Theorem 1.2.3. *The Julia set for any rational map of degree $d \geq 2$ is equal to the closure of its set of repelling periodic points.*

1.3 Polynomial-like maps

The notion of polynomial-like mappings was introduced by Douady and Hubbard in the landmark paper *On the dynamics of Polynomial-like mappings* ([DH]). The dynamics of polynomials and notably quadratic polynomials was the first object of study in the field of holomorphic dynamics, because half of the dynamics of such systems is tame and gives a platform for studying the complicated dynamics of the remaining half. Polynomial-like mappings has proven to be instrumental in the understanding and solving a host of problems in holomorphic dynamics: it provides a language for formulating the notion of renormalization of polynomials and other holomorphic maps, it is essential in the description of the locus of cubic polynomials with at least one escaping critical points by Branner and Hubbard ([BH]), etc.

Definition 1.3.1. A *polynomial-like map* of degree $d \geq 2$ is a triple (f, U, U') where U, U' are open sets of $\mathbb{C}$ isomorphic to discs with $\overline{U'} \subset U$ and $f : U' \to U$ is a proper holomorphic map of degree d.

The filled Julia set and the Julia set are defined for polynomial-like maps in the same fashion as for polynomials.

Definition 1.3.2. Let $f : U' \to U$ be a polynomial-like map. The *filled Julia set* of f is defined as the set of points in U' that never leave U' under iteration, i.e.

$$K_f = \{z \in U' \mid f^n(z) \in U' \; \forall n \geq 0\}$$

An equivalent definition is

$$K_f = \bigcap_{n \geq 0} f^{-n}(\overline{U'})$$

and from this expression it is clear that K_f is a compact set.

As for polynomials, we define the Julia set of f as

$$J_f := \partial K_f$$

Any polynomial-like map (f, U', U) of degree d is associated with an *external map* h_f of the same degree d, which describes the dynamics of the polynomial-like map outside the filled Julia set. We will give the construction of an external class for polynomial-like maps in the case K_f is connected. For the case K_f not connected, we refer to [DH].

Let (f, U', U) be a polynomial-like map of degree d with connected filled Julia set K_f. Let

$$\alpha : U \setminus K_f \longrightarrow W = \{z | 1 < |z| < R\}$$

(where $logR$ is the modulus of $U \setminus K_f$) be an isomorphism such that $|\alpha(z)| \to 1$ as $z \to K_f$. Write $W' = \alpha(U' \setminus K_f)$ and define the map:

$$h^+ := \alpha \circ f \circ \alpha^{-1} : \; W' \to W.$$

Since the filled Julia set is connected, it contains all the critical points of f, then $f : U' \setminus K_f \to U \setminus K_f$ is a holomorphic degree d covering map, therefore the map h^+ is a holomorphic degree d covering. Let $\tau(z) = 1/\bar{z}$ be the reflection with respect to the unit circle, and set $W_- = \tau(W)$, $W'_- = \tau(W')$, $\widetilde{W} = W \cup \mathbb{S}^1 \cup W_-$ and $\widetilde{W'} = W' \cup \mathbb{S}^1 \cup W'_-$. We can extend analytically the map $h^+ : W' \to W$ to an analytic map $h : \widetilde{W'} \to \widetilde{W}$ by the strong reflection principle with respect to $\mathbb{S}^1$. The mapping is strictly expanding. Indeed $h : \widetilde{W'} \to \widetilde{W}$ is a degree d covering map, and $h^{-1} : \widetilde{W} \to \widetilde{W'} \subsetneq \widetilde{W}$ is strongly contracting for the Poincare metric on $\widetilde{W}$. Let h_f be the restriction of h to the unit circle. Then the map $h_f : \mathbb{S}^1 \to \mathbb{S}^1$ is an *external map* of f.

It is easy to see (by theorem 1.1.6) that the external map of a polynomial of degree d is $z \to z^d$. The next theorem shows that a polynomial-like map of degree d with external map $z \to z^d$ is equivalent to a polynomial of degree d.

Theorem 1.3.3. (Straightening theorem), *Let $f : U' \to U$ be a polynomial-like map of degree d. Then, there exists a polynomial P of degree d and a quasiconformal map φ such that*

$$f = \varphi \circ P \circ \varphi^{-1}$$

on U'. Moreover, if K_f is connected, then P is unique up to (global) conjugation by an affine map.

Proof. The idea of the proof is to replace the external map of a polynomial-like map with the external map of a polynomial, i.e. $P_d(z) := z \to z^d$, and then to prove that a polynomial-like map of degree d with external map P_d is equivalent to a polynomial of degree d. We will not prove the unicity here.

Let us assume U and U' with smooth boundaries. Define $Q_f = U \setminus U'$, then Q_f is a topological annulus. Set $B = \mathbb{D}_{R^d}$, where $R > 1$ and $d =$degree$f$. Set $B' = \mathbb{D}_R = P_d^{-1}(B)$. Then $P_d : B' \setminus \overline{\mathbb{D}} \to B \setminus \overline{\mathbb{D}}$ is a degree d covering map. Define $Q_B = B \setminus B'$

Let $\psi_0 : \partial U \to \partial B$ be an orientation-preserving C^1-diffeomorphism, let $\psi_1 : \partial U' \to \partial B'$ be a lift of $\psi_0 \circ f$ with respect to P_d. Define a quasiconformal map $\psi : \overline{Q}_f \to \overline{Q}_B$ as follows:

$$\psi(z) = \begin{cases} \psi_0 & \text{on } \partial U \\ \psi_1 & \text{on } \partial U' \\ \text{quasiconformal interpolation} & \text{on } Q_f \end{cases}$$

Define on U an almost complex structure μ as follows:

$$\mu(z) = \begin{cases} \bar{\mu} = \psi^*(\mu_0) & \text{on } Q_f \\ (f^n)^*\bar{\mu} & \text{on } f^{-n}(Q_f) \\ \mu_0 & \text{on } K_f \end{cases}$$

Then μ is bounded since ψ is quasiconformal and f is holomorphic, and it is f-invariant by construction. Thus, by the Ahlfors, Bers, Morrey, Bojarski Measurable Riemann Mapping Theorem there exists $\varphi : U \to \mathbb{D}$ such that $\varphi^*\mu_0 = \mu$. Set $V = \varphi(U),\ V' = \varphi(U')$. Hence $g = \varphi \circ f \circ \varphi^{-1} : V' \to V$ is a polynomial-like map of degree d, hybrid conjugate to f and with external class P_d.

Let S be the Riemann surface obtained by gluing V and $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, by the equivalence relation identifying z to $\psi(z)$, i.e.

$$S = (V) \coprod (\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})/z \sim \varphi(z).$$

Then S is isomorphic to the Riemann sphere, by the Uniformization theorem. Define the map $\widetilde{g}$ as follows:

$$\widetilde{g}(z) = \begin{cases} g & \text{on } V' \\ P_d & \text{on } \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \end{cases}$$

Since the map P_d is the external map of g, the map $\tilde{g}$ is continuous and then holomorphic. Let $\widehat{\phi} : S \to \widehat{\mathbb{C}}$ be an isomorphism that fixes infinity. Define $P = \widehat{\phi} \circ \tilde{g} \circ \widehat{\phi}^{-1} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. The map P is a holomorphic function hybrid conjugate to the map f. Since $P^{-1}(\infty) = (\widehat{\phi} \circ \tilde{g} \circ \widehat{\phi}^{-1})^{-1}(\infty) = \widehat{\phi} \circ \tilde{g}^{-1} \circ \widehat{\phi}^{-1}(\infty) = \infty$ (since $\widetilde{g}$ outside V' is a polynomial), then P is a polynomial.

We refer the reader to [DH] for the proof of uniqueness.

□

Now, let $\mathbf{f} = (f_\lambda : U'_\lambda \to U_\lambda)_{\lambda \in \Lambda}$, where $\Lambda \approx \mathbb{D}$ be a family of polynomial-like mappings. Define **U'** $= \{(\lambda, z) |\ z \in U'_\lambda\}$, $\mathbf{U} = \{(\lambda, z) |\ z \in U_\lambda\}$, and $f(\lambda, z) = (\lambda,\ f_\lambda(z))$. Then **f** is an *analytic family of polynomial-like mappings* if the following conditions are satisfied:

1. **U'** and **U** are homeomorphic over Λ to $\Lambda \times \mathbb{D}$;
2. the projection from the closure of **U'** in **U** to Λ is proper;
3. the map $f : \mathbf{U'} \to \mathbf{U}$ is complex analytic and proper.

The degree of the family f_λ is independent of λ. Set $K_\lambda = K_{f_\lambda}$, $J_\lambda = J_{f_\lambda}$ and define

$$M_f = \{\lambda \mid K_\lambda \text{ is connected}\}.$$

By the Straightening theorem, for every $\lambda \in \Lambda$ the map f_λ is hybrid equivalent to a polynomial, and if K_λ is connected this polynomial is unique. Hence in degree 2 we can define a map:

$$\chi : M_f \to M$$

$$\lambda \to c,$$

which associates to every $\lambda \in M_f$ the $c \in M$ such that f_λ is hybrid equivalent to $P_c = z^2 + c$.

Let c_λ be the critical point of f_λ. Suppose there exists $A \subset \Lambda$ such that $f_\lambda(c_\lambda) \in U_\lambda \setminus U'_\lambda$ for $\lambda \in \Lambda \setminus A$. Then M_f is compact. In [DH] is proven that:

Theorem 1.3.4. *Suppose M_f compact. Let $A \subset \Lambda$ be a subset homeomorphic to $\overline{\mathbb{D}}$ such that $M_f \subset \mathring{A}$. Then the map $\chi : \Lambda \to \mathbb{C}$ is a branched covering of degree $\mathcal{D}$ equal to the number of times $f_\lambda(c_\lambda) - c_\lambda$ turns around 0 as λ describes ∂A. Moreover, if $\mathcal{D} = 1$, M_f is a quasiconformal copy of M.*

Chapter 2

Parabolic-like mappings

2.1 Introduction

A polynomial-like map of degree d is a triple (f, U', U) where U', U are open subsets of $\mathbb{C}$, $U', U \approx \mathbb{D}$, $U' \subset\subset U$, and $f : U' \to U$ is a proper holomorphic map of degree d. These were originally singled out and studied by Douady and Hubbard in the groundbreaking paper *On the Dynamics of Polynomial-like Mappings*, see [DH]. A polynomial-like map of degree d is determined up to holomorphic conjugacy by its internal and external classes. In particular the external class is a degree d real-analytic orientation preserving and strictly expanding self-covering of the unit circle. Note that the expansivity of such a circle map implies that all the periodic points are repelling, and in particular not parabolic.

The aim of this thesis is, in some sense, to avoid this restriction. More precisely we will define an object, a *parabolic-like mapping*, to describe the parabolic case. A parabolic-like mapping is thus similar to a polynomial-like mapping, but with a parabolic external class, *i.e.* an external map with a parabolic fixed point. This implies that the domain is not contained in the codomain.

Let $Per_1(1)$ be the set of Möbius conjugacy classes of quadratic rational maps with a parabolic fixed point of multiplier 1. If we fix the parabolic fixed point to be infinity and the critical points to be ± 1, then we obtain

$$Per_1(1) = \{[P_A] \mid P_A(z) = z + 1/z + A,\ A \in \mathbb{C}\}.$$

By analogy with the theory of polynomial-like mappings, we prove a Straightening Theorem for parabolic-like maps, which states that any parabolic-like map of degree 2 is hybrid conjugate to a representative of a class in $Per_1(1)$, a unique such class if the filled Julia set is connected.

The maps belonging to the conjugacy classes of $Per_1(1)$ have two simple critical points at $z = \pm 1$, and, for $A \neq 0$, a parabolic fixed point at infinity and another fixed point at $z = -\frac{1}{A}$. For $A = 0$ we obtain the map $P_0(z) = z + 1/z$, which has just one fixed point which is a double parabolic fixed point at infinity. This map is conformally equivalent to the map $h_2 = \frac{3z^2+1}{3+z^2}$ under the Möbius transformation which sends $z = 1$ to infinity, $z = -1$ to $z = 0$ and infinity to $z = 1$. The other maps P_A, with $A \neq 0$, are not globally conformally conjugate to the map h_2, but we prove they are still conjugate to h_2 outside their filled Julia set if it is connected, or on part of the basin of infinity if not. Therefore the map h_2 is the *external map* of the family P_A (see Prop. 2.5.1).

In this chapter we will first define a parabolic-like map and the filled Julia set of a parabolic-like map. Then we will construct and discuss the external class in this extended setting. Finally, the Straightening Theorem for parabolic-like maps will be obtained by replacing its external class by that of h_2.

2.2 Definitions

For a parabolic-like mapping, the set of points with infinite forward orbit is not contained in the intersection of the domain and the range. This calls for a partition of this set into a filled Julia set compactly contained in both domain and range and exterior attracting petals.

Definition 2.2.1. (Parabolic-like maps) A parabolic-like map of degree d is a 4-tuple (f, U', U, γ) where

- U', U are open subsets of $\mathbb{C}$, with U', U and $U \cup U'$ isomorphic to a disc, and U' not contained into U,
- $f : U' \to U$ is a proper holomorphic map of degree d with a parabolic fixed point at $z = z_0$ of multiplier 1,
- $\gamma : [-1, 1] \to \overline{U}, \gamma(0) = z_0$ is an arc, forward invariant under f, C^1 on $[-1, 0]$ and on $[0, 1]$, and such that

$$f(\gamma(t)) = \gamma(dt), \ \ \forall -\frac{1}{d} \leq t \leq \frac{1}{d},$$

$$\gamma([\frac{1}{d}, 1) \cup (-1, -\frac{1}{d}]) \subseteq U \setminus U',$$

$$\gamma(\pm 1) \in \partial U.$$

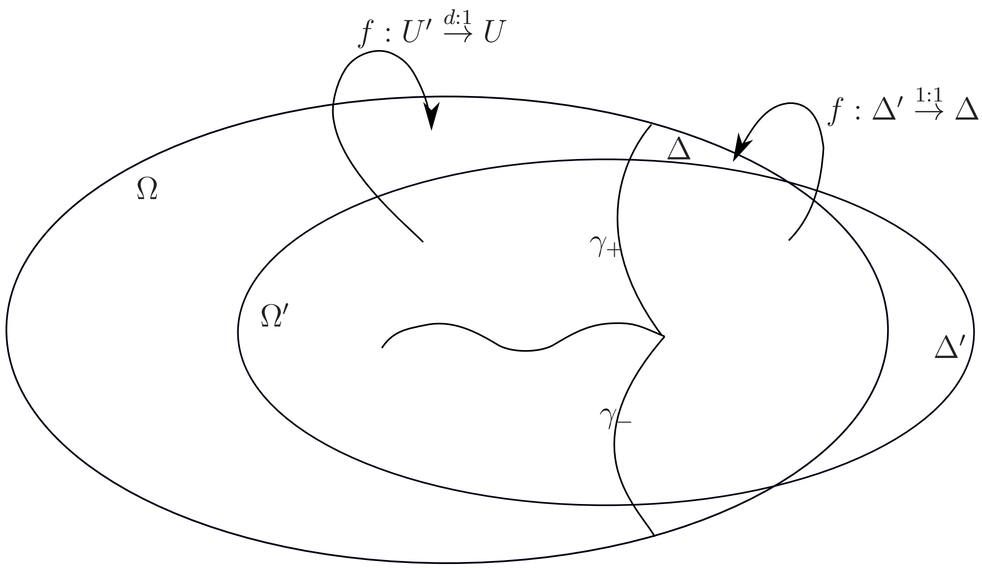


Figure 2.1: On a parabolic-like map (f, U', U, γ) the arc γ divides U', U into Ω', Δ' and Ω, Δ respectively. These sets are such that Ω' is compactly contained in U, $\Omega' \subset \Omega$ and $f : \Delta' \to \Delta$ is an isomorphism.

It resides in repelling petal(s) of z_0 and it divides U', U into Ω', Δ' and Ω, Δ respectively, such that $\Omega' \subset\subset U$ (and $\Omega' \subset \Omega$), $f : \Delta' \to \Delta$ is an isomorphism (see Fig. 2.1) and Δ' contains at least one attracting fixed petal of z_0. We call the arc γ a *dividing arc*.

Notation. *We can consider $\gamma := \gamma_+ \cup \gamma_-$, where $\gamma_+(t) = \gamma(t), t \in [0,1]$, and $\gamma_-(t) = \gamma(-t), t \in [0,1]$ (i.e. $\gamma_+ : [0,1] \to \overline{U}$, $\gamma_- : [0,-1] \to \overline{U}$, $\gamma_\pm(0) = z_0$). Where it will be convenient (e.g. in the examples) we will refer to $\gamma_\pm$ instead of γ. Therefore we will often consider a parabolic-like map as a 5-tuple $(f, U', U, \gamma_+, \gamma_-)$ instead of a 4-tuple (f, U', U, γ). These two notions are equivalent.*

The filled Julia set and the Julia set are defined for parabolic-like maps in the same fashion as for polynomials.

Definition 2.2.2. Let (f, U', U, γ) be a parabolic-like map. We define the *filled Julia set* K_f of f as the set of points in U' that never leave $(\Omega' \cup \gamma_\pm(0))$ under iteration, i.e.

$$K_f = \{z \in U' \mid \forall n \geq 0 \ , \ f^n(z) \in \Omega' \cup \gamma_\pm(0)\}.$$

Motivations for the definition

A parabolic-like map can be seen as the union of two different dynamical parts: a polynomial-like part (on Ω') and a parabolic one (on Δ'), which are connected by the dividing arc γ. Indeed, even if the arc can be *constructed* a posteriori by Fatou coordinates since it resides in repelling petal(s), we *define* it to ensure the existence of these two different parts, thus to separate the filled Julia set from the exterior attracting petal(s). This moreover guarantees the existence of an annulus, $U \setminus \Omega'$, essential to perform the surgery which will give the Straightening Theorem.

We take as domain of a parabolic-like map a topological disc U' *containing* the parabolic fixed point to insure the filled Julia set to be *compactly contained* in the intesection of the domain and the range, and thus to define an *external map*.

There are many prospect definitions of a parabolic-like map. The one introduced here is flexible enough to capture many interesting examples, and rigid enough to allow for a viable theory.

Remark 2.2.1. *An equivalent definition for the filled Julia set of f is*

$$K_f = \bigcap_{n \geq 0} f^{-n}(U \setminus \Delta).$$

The filled Julia set is a compact subset of $U \cap U'$ and, if it is connected, it is full (since it is the intersection of topological disks).

As for polynomials, we define the Julia set of f as

$$J_f := \partial K_f$$

2.2.1 Examples

1. Consider the function $h_2(z) = \frac{3z^2+1}{3+z^2}$. This map has a parabolic fixed point at $z = 1$ of parabolic multiplicity 2 and multiplier 1, and critical points at $z = 0$ and at ∞. Note that the two critical points are in different components of the immediate parabolic basin of attraction.

   Choose $\epsilon > 0$ and define $U' = \{z : |z| < 1 + \epsilon\}$, and $U = h_2(U')$. Since the parabolic fixed point has multiplicity 2, there are 4 petals (2 repelling petals and 2 attracting ones, alternating) whose union form a neighborhood of the parabolic fixed point $z = 1$. The attracting directions of the parabolic fixed point are along the real axis, while the repelling ones are perpendicular to the real axis. Let $\Xi_\pm$ be the repelling petals. The repelling petals $\Xi_\pm$ intersect the unit circle and can be taken to be reflection symmetric around the unit circle, since h_2 is autoconjugate by the reflection $T(z) = \frac{1}{\bar{z}}$. Let $\phi_\pm : \Xi_\pm \to \mathbb{H}_-$ be Fatou coordinates with axis tangent to the unit circle at the parabolic fixed point $z = 1$. The image of the unit circle in the Fatou coordinate planes are horizontal lines that can be normalized to be the negative real axis. Choose $m > 0$ such that

$$\forall z \in \phi_+(\mathbb{S}^1 \cap \Xi_+) \ \ \mathrm{Im}(z) > -m$$

$$\forall z \in \phi_-(\mathbb{S}^1 \cap \Xi_-) \ \ \mathrm{Im}(z) < m$$

and define the dividing arcs as:

$$\gamma_+ := \phi_+^{-1}(-mi + \mathbb{R}_-) : \mathbb{C} \to \Xi_+,$$
$$\gamma_- := \phi_-^{-1}(mi + \mathbb{R}_-) : \mathbb{C} \to \Xi_-.$$

In order to obtain

$$h_2(\gamma_\pm(t)) = \gamma_\pm(dt) \ \ \forall 0 \leq \pm t \leq \frac{1}{d}$$

and $\gamma_+ : [0,1] \to \Xi_+, \ \ \gamma_- : [0,-1] \to \Xi_-$,

we need to reparametrize the arcs. Since

$$exp \circ Re : \mathbb{C} \to \mathbb{R}_- \to [0,1]$$

$$z \to exp(Re(z))$$

and

$$-exp \circ Re : \mathbb{C} \to \mathbb{R}_- \to [0,-1]$$

$$z \to -exp(Re(z))$$

let us consider

$$\gamma_+ : [0,1] \to \Xi_+$$
$$t \to \phi_+^{-1}(log_d(t) - im),$$

and

$$\gamma_- : [0,-1] \to \Xi_-$$
$$t \to \phi_-^{-1}(log_d(-t) + im).$$

Then $(h_2, U', U, \gamma_\pm)$ is a parabolic-like map of degree 2.

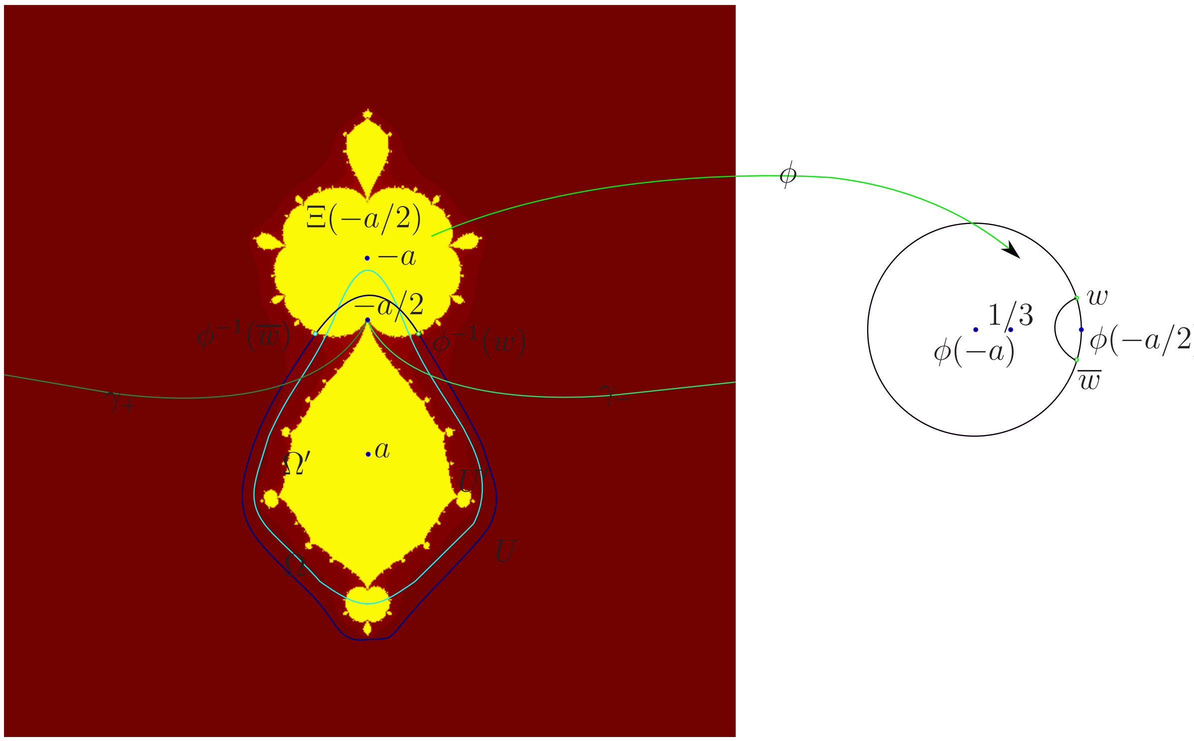


Figure 2.2: Construction of a parabolic-like restriction of the map $f = z^3 - 3a^2 z + 2a^3 + a$, for $a = -2/3i$.

2. Let $f(z) = z^3 - 3a^2 z + 2a^3 + a$, for $a = -2/3i$. This map has a superattracting fixed point at $z = a$, a parabolic fixed point at $z = -a/2$ with multiplier and parabolic multiplicity 1 and a critical point at $z = -a$. Call $\Xi(-a/2)$ the immediate basin of attraction of the parabolic fixed point. Then the critical point $z = -a$ belongs to $\Xi(-a/2)$. Let $\phi : \Xi(-a/2) \to \mathbb{D}$ be the Riemann map normalized by setting $\phi(-a) = 0$ and $\phi(z) \overset{z \to -a/2}{\longrightarrow} 1$, and let $\psi : \mathbb{D} \to \Xi(-a/2)$ be its inverse. By the Carathodory theorem the map ψ extends continuously to $\mathbb{S}^1$. Note that $\phi \circ f \circ \psi = h_2$. Let w be an h_2 periodic point in the first quadrant, such that the hyperbolic geodesic $\widetilde{\gamma} \in \mathbb{D}$ connecting w and $\overline{w}$ separates the critical value $z = 1/3$ from the parabolic fixed point $z = 1$. Let U be the Jordan domain bounded by $\widehat{\gamma} = \psi(\widetilde{\gamma})$, union the arcs up to potential level 1 of the external rays landing at $\psi(w)$ and $\psi(\overline{w})$, together with the arc of the level 1 equipotential connecting this two rays around $z = a$ (see Fig. 2.2). Let U' be the preimage of U under f and the dividing arcs $\gamma_{\pm}$ be the fixed external rays landing at the parabolic fixed point $z = 1/3i$ and parametrized by potential.

Then $(f, U', U, \gamma_{\pm})$ is a parabolic-like map of degree 2 (see Fig. 2.3).

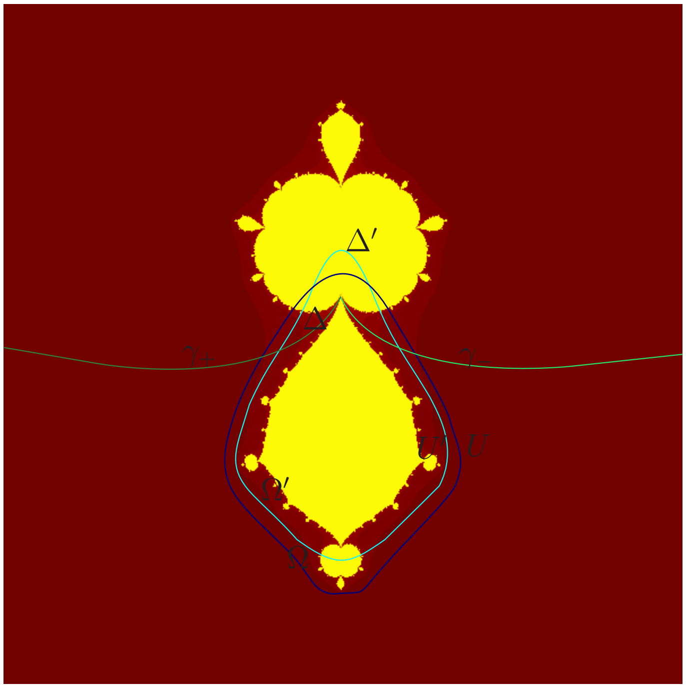


Figure 2.3: A parabolic-like restriction of the map $f_a = z^3 - 3a^2 z + 2a^3 + a$, for $a = -2/3i$.

3. Let $f(z) = z^2 + c$, for $c = (-1 + i\sqrt{3})/8$ (fat rabbit). Its third iterate f^3 has a parabolic fixed point $a = (-1 + i\sqrt{3})/4$ of multiplier 1 and parabolic multiplicity 3.
   Let Ξ_0 be the component containing $z = 0$ of the immediate basin of attraction of the parabolic fixed point. Number the connected components of the immediate attracting basin in the dynamical order (which here is the counterclockwise direction around a). Let $\phi : \Xi_0 \to \mathbb{D}$ be the Riemann map, normalized by $\phi(0) = 0$ and $\phi(z) \overset{z \to a}{\longrightarrow} 1$, and let $\psi : \mathbb{D} \to \Xi_0$ be its inverse. The map ψ extends continuously to $\mathbb{S}^1$, and $\phi \circ f^3 \circ \psi = h_2$. As above let w be a h_2 periodic point in the first quadrant such that the hyperbolic geodesic $\widetilde{\gamma}$ connecting w and $\overline{w}$ separates the critical value $z = 1/3$ from the parabolic fixed point $z = 1$. Define $\widehat{\gamma} = \psi(\widetilde{\gamma})$ and $\widehat{\gamma'} = f^{-1}(\widehat{\gamma}) \cap \overline{\Xi_2}$. Let U be the Jordan domain bounded by $\widehat{\gamma}$ union the arcs up to potential level 1 of the external rays landing at $\psi(w)$ and $\psi(\overline{w})$ union $\widehat{\gamma'}$ union the arcs up to potential level 1 of the external rays landing at $f^{-1}(\psi(w)) \cap \overline{\Xi_2}$ and $f^{-1}(\psi(\widetilde{w})) \cap \overline{\Xi_2}$, together with the two arcs of the level 1 equipotential connecting this

four rays around the parabolic fixed point. Let $U' \subset\subset U$ be the preimage of U under f^3 and the dividing arcs γ_+, γ_- be the external rays for angles 1/7 and 2/7 respectively parametrized by potential. Then $(f^3, U', U, \gamma_\pm)$ is a parabolic-like map of degree 2 (see Fig. 2.4).

More generally, define $\lambda_{p/q} = \exp(2\pi i p/q)$ with p and q coprime, $c_{p/q} = \frac{\lambda_{p/q}}{2} - \frac{\lambda^2_{p/q}}{4}$ and consider $f_q = z^2 + c_{p/q}$. The map f_q has a parabolic fixed point of multiplier $\lambda_{p/q}$ at $a = \lambda_{p/q}/2$, therefore f^q has a parabolic fixed point a of multiplier 1 and parabolic multiplicity q.

Repeating the construction done above one can see that f^q presents a degree 2 parabolic-like restriction.

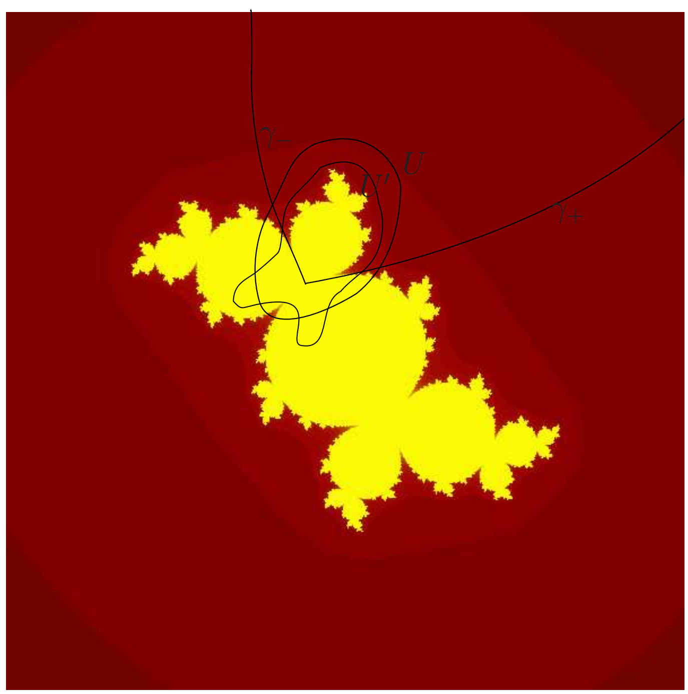


Figure 2.4: A parabolic-like restriction of the third iterate of the map $f = z^2 + c$, for $c = -0.125 + 0.6495i$.

As we can see from the examples, there are many different equivalent choices for the domain and codomain of a parabolic-like map. This is because the notion of parabolic-like map (as well as the notion of polynomial-like map) is local.

Definition 2.2.3. Let (f, U', U, γ) be a parabolic-like map of degree d and filled Julia set K_f. We say that (f, V', V, γ_s) is a *parabolic-like restriction* of (f, U', U, γ) if $V' \subseteq U'$ and (f, V', V, γ_s) is a parabolic-like map with the same degree and filled Julia set of (f, U', U, γ).

Note that, trivially, every parabolic-like map is a parabolic-like restriction of itself.

Definition 2.2.4. Let (f, U', U, γ) be a parabolic-like map of degree d, and let $\gamma_s : [-1, 1] \to \overline{U}$ be an arc forward invariant under f and such that $\gamma_s(0) = z_0$ (where z_0 is the parabolic fixed point of f). We say that γ and γ_s are *isotopic/equivalent* if there exists $V' \subseteq U'$ such that (f, U', U, γ) and (f, V', V, γ_s) have a *common parabolic-like restriction* (see also Lemma 2.2.1).

Note that, if (f, U', U, γ) is a parabolic-like map, and γ_s is isotopic to γ, (f, U', U, γ_s) might not be a parabolic-like map. Indeed, we do not ask $\gamma_{s+}(1) \in \partial U$ and $\gamma_{s-}(-1) \in \partial U$. On the other hand, there exists $V' \subseteq U'$ such that (f, V', V, γ_s) is a parabolic-like restriction of (f, U', U, γ) (and, trivially, of itself). Hence (f, V', V, γ_s) and (f, U', U, γ) are parabolic-like maps with same degree and filled Julia set.

Definition 2.2.5. Let (f, U', U, γ) and (f, V', V, γ_s) be parabolic-like maps of the same degree d. We say that (f, U', U, γ) and (f, V', V, γ_s) are *equivalent* if they have a common parabolic-like restriction. If (f, U', U, γ) and (f, V', V, γ_s) are equivalent *we do not distinguish between them.*

Note that, if (f, V', V, γ_s) is a parabolic-like restriction of (f, U', U, γ), then (f, V', V, γ_s) and (f, U', U, γ) are equivalent. Similarly, if (f, U', U, γ) is a parabolic-like map, and γ_s is isotopic to γ, there exists $V \subseteq U$ such that (f, V', V, γ_s) and (f, U', U, γ) are equivalent. In particular, if $V = U$, (f, U', U, γ_s) and (f, U', U, γ) are equivalent. Hence, the dividing arc of a parabolic-like map *is defined up to isotopy.*

Lemma 2.2.1. *Let (f, U', U, γ) be a parabolic-like map, and let γ_s be isotopic to γ. Then the projections of γ_s and γ to Ecalle cylinders are isotopic modulo the projections of K_f and critical points.*

On the othe hand, let (f, U', U, γ) be a parabolic-like map and let $\gamma_s : [-1, 1] \to \overline{U}$ be an arc forward invariant under f, with $\gamma_s(0) = z_0$ (where z_0 is the parabolic fixed point of f). If the projections of γ_s and γ to Ecalle cylinders are isotopic modulo the projections of K_f and critical points, γ_s is isotopic to γ.

Proof. The first implication is trivial, let us prove the second one.

Let Ξ_+ and Ξ_- be the repelling petals where γ_+ and γ_- respectively reside (note that the parabolic fixed point z_0 for f may have parabolic multiplicity

1, hence just one attracting and one repelling petal. In this case Ξ_+ and Ξ_- coincide). Then the quotient manifolds Ξ_+/f, Ξ_-/f are conformally isomorphic to the bi-infinite cylinder, i.e. $\Xi_+/f \approx \mathbb{C}/\mathbb{Z}$, $\Xi_-/f \approx \mathbb{C}/\mathbb{Z}$. Call β the isomorphism between Ξ_+/f and $\mathbb{C}/\mathbb{Z}$, and δ the isomorphism between Ξ_-/f and $\mathbb{C}/\mathbb{Z}$. Let

$$H_+ : [0,1] \times \mathbb{R}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$$

$$(s,t) \to H_+(s,t),$$

$$H_- : [0,1] \times \mathbb{R}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$$

$$(s,t) \to H_-(s,t),$$

be isotopies, disjoint from the projection of the filled Julia set and the critical points, such that for every fixed $s \in [0,1]$, both $H_\pm(s,t) : \mathbb{R}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ are at least C^1. Set $\gamma_{s+}[\tau, d\tau] = \beta^{-1}(H_+(s,\cdot))$ and $\gamma_{s-}[d\hat{\tau}, \hat{\tau}] = \delta^{-1}(H_-(s,\cdot))$ Define γ_s by extending γ_{s+} and γ_{s-} by the dynamics of f to forward invariant curves in Ξ_+, Ξ_- respectively (see Picture 2.5), i.e.:

1. $\gamma_{s+}(d^n t) = f^n(\gamma_{s+}(t)),\ \gamma_{s+}(t/d^n) = f(\gamma_{s+}(t))^{-n}\ \ \forall \tau \leq t \leq d\tau$;

2. $\gamma_{s-}(d^n t) = f^n(\gamma_{s-}(t)),\ \gamma_{s-}(t/d^n) = f(\gamma_{s-}(t))^{-n}\ \ \forall d\hat{\tau} \leq t \leq \hat{\tau}$;

3. $\gamma_s(\pm 1) \in \partial U$;

4. and $\gamma_s(0) = z_0$;

where $f(\gamma_s)^{-n}$ is the branch which gives continuity. Then (f, U', U, γ_s) and (f, U', U, γ) have a common parabolic-like restriction. Indeed, γ_s divides U and U' in Ω_s, Δ_s and Ω'_s, Δ'_s respectively, and since the projections of γ_s and γ to Ecalle cylinders are isotopic modulo the projections of K_f and critical points, Ω'_s contains K_f and all the critical points of (f, U', U, γ). Hence (f, U', U, γ_s) is a parabolic-like map with the same degree and filled Julia set as (f, U', U, γ), and thus it is a parabolic-like restriction of (f, U', U, γ) (and trivially of itself). Therefore, the arcs γ and γ_s are isotopic.

□

Note that, by construction, if (f, U', U, γ) is a parabolic-like map and γ_s is an equivalent dividing arc, then the arc γ_{+s} resides in the same petal as γ_+ and the arc γ_{-s} resides in the same petal as γ_-.

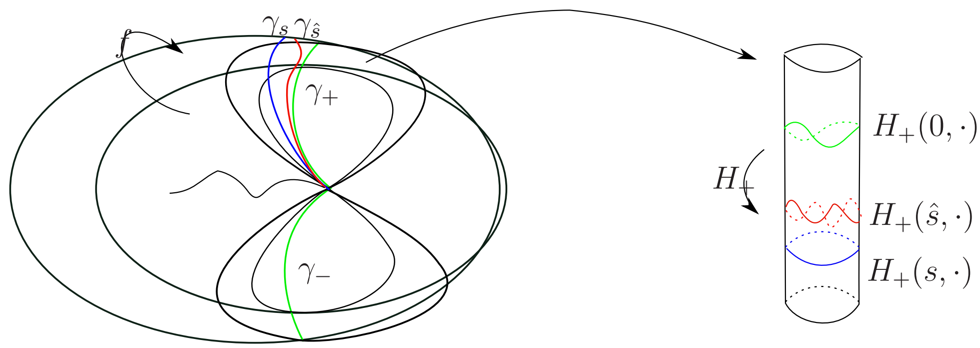


Figure 2.5: Construction of dividing arcs equivalent to γ.

2.3 The external class of f

In analogy with the polynomial-like setting, we want to associate to any parabolic-like map (f, U', U, γ) of degree d a real-analytic map $h_f : \mathbb{S}^1 \to \mathbb{S}^1$ of the same degree d and with a parabolic fixed point, unique up to conjugacy by a real-analytic diffeomorphism. We will call h_f an *external map* of f, and we will call $[h_f]$ (its conjugacy class under analytic diffeomorphism) the external class of f.

2.3.1 The construction of an external map of a parabolic-like map f with connected Julia set

The construction of an external map of a parabolic-like map with connected Julia set follows the construction of an external map in [DH], up to the differences given by the geometry of our setting.

Let (f, U', U, γ) be a parabolic-like map of degree d with connected filled Julia set K_f. Then K_f contains all the critical points of f and hence $f : U' \setminus K_f \to U \setminus K_f$ is a holomorphic degree d covering map. Let

$$\alpha : \overline{\mathbb{C}} \setminus K_f \longrightarrow \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$$

be the Riemann map, normalized by $\alpha(\infty) = \infty$ and $\alpha(\gamma(t)) \to 1$ as $t \to 0$. Write $W' = \alpha(U' \setminus K_f)$ and $W = \alpha(U \setminus K_f)$ (see Fig. 2.6) and define the map:

$$h^+ := \alpha \circ f \circ \alpha^{-1} : \; W' \to W,$$

Then the map h^+ is a holomorphic degree d covering. Let $\tau(z) = 1/\bar{z}$ denote the reflection with respect to the unit circle, and define $W_- = \tau(W)$, $W'_- = \tau(W')$, $\widetilde{W} = W \cup \mathbb{S}^1 \cup W_-$ and $\widetilde{W'} = W' \cup \mathbb{S}^1 \cup W'_-$. Applying the strong

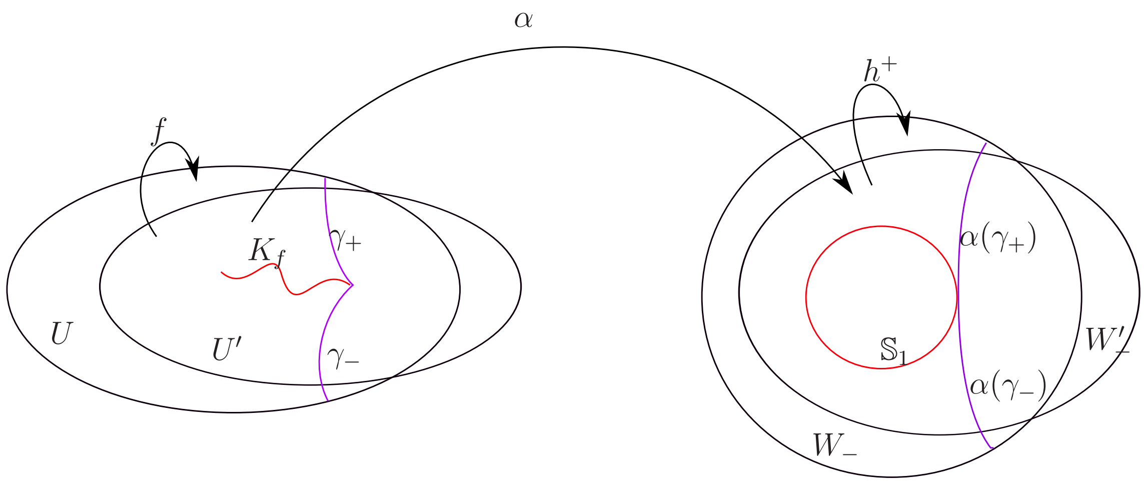
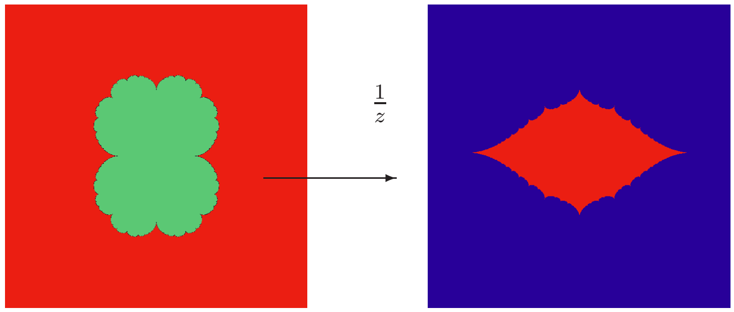


Figure 2.6: Construction of an external map in the case K_f connected. We set $W' = \alpha(U' \setminus K_f),\ W = \alpha(U \setminus K_f)$ and $h^+ : W' \to W$.

reflection principle with respect to $\mathbb{S}^1$ we can extend analytically the map $h^+ : W' \to W$ to $h : \widetilde{W'} \to \widetilde{W}$. Let h_f be the restriction of h to the unit circle, then the map $h_f : \mathbb{S}^1 \to \mathbb{S}^1$ is an *external map* of f. A parabolic external map is defined up to real-analytic diffeomorphism.

Remark 2.3.1. *As we have seen, we can construct a canonical external map of f when K_f is connected. Therefore in the case K_f connected we could speak about 'the' external map of f, instead of 'an' external map. However we prefer to refer to this map as 'an' external map of f and to consider more gererically the external 'class' of f in order to allow more flexibility to our setting.*

Note that if (f, U', U, γ) is a parabolic-like map, then there exists at least one attracting fixed petal outside the filled Julia set K_f. Indeed the external map $h_f : \mathbb{S}^1 \to \mathbb{S}^1$ has a parabolic-fixed point if and only if there exists at least one attracting fixed petal outside the filled Julia set K_f. Consider for example the cauliflower $f(z) = z^2 + 1/4$. This map has a parabolic fixed point at $z = 1/2$ of parabolic multiplicity and multiplier 1, but it cannot present a parabolic-like restriction. Indeed the parabolic basin of attraction resides in the interior of the filled Julia set, while the repelling direction resides on the Julia set and outside of it. Therefore its external map is hyperbolic. On the other hand, conjugating the cauliflower with the inversion $\iota(z) = 1/z$ we obtain the map $f(z) = \frac{4z^2}{4+z^2}$, which presents a parabolic-like restriction.



2.3.2 The general case

Let (f, U', U, γ) be a parabolic-like map of degree d. To deal with the case where the filled Julia set is not connected, we will lean on the similar construction in the polynomial-like case. We construct annular Riemann surfaces T and T' that will play the role of $U' \setminus K_f$ and $U \setminus K_f$ respectively, and an analytic map $F : T \to T'$ that will play the role of f.

Let $V \approx \mathbb{D}$ be a full relatively compact connected subset of U containing $\overline{\Omega}'$ and the critical values of f and such that $f : f^{-1}(V) \to V$ is a parabolic-like restriction of (f, U', U, γ).

Let us call $L = f^{-1}(\overline{V}) \cap \overline{\Omega}'$ and $M = f^{-1}(\overline{V}) \cap \Delta'$. Define $X'_0 = (U \cup U') \setminus L$, $U_0 = U \setminus \overline{V}$, $A_0 = U \cap U' \setminus L$, $X_0 = U \setminus L$, $A'_0 = U' \setminus L$ and $A''_0 = U' \setminus f^{-1}(\overline{V})$. Note that X_0 is an annular domain.

Let $\rho_0 : X_1 \to X_0$ be a degree d covering map for some Riemann surface X_1, and define $V_1 = \rho_0^{-1}(V \setminus L)$. Define $X''_1 = X_1 \setminus \overline{V_1}$. The map $f : A''_0 \to U_0$ is proper holomorphic of degree d, and $\rho_0 : X''_1 \to U_0$ is a proper holomorphic map of degree d. Therefore we can choose $\pi_0 : A''_0 \to X''_1$, a lift of $f : A''_0 \to U_0$ to $\rho_0 : X''_1 \to U_0$, and π_0 is an isomorphism. The subset Δ has d preimages under the map ρ_0. Let us call Δ_1 the preimage of Δ under ρ_0 such that $\Delta_1 \cap \pi_0(A''_0 \cap \Delta') \neq \varnothing$. Since $f : \Delta' \to \Delta$ is an isomorphism, we can extend the map π_0 to Δ'. Let us call $B'_1 = X''_1 \cup \Delta_1$. Since $\pi_0(\Delta' \setminus A''_0) \cap X''_1 = \varnothing$, the extension $\pi_0 : A'_0 \to B'_1$ is an isomorphism (see Fig 2.7). Let us call $B_1 = \pi_0(A_0)$. Define $A'_1 = \rho_0^{-1}(A_0)$ and $f_1 = \pi_0 \circ \rho_0 : A'_1 \to B_1$. The map f_1 is proper, holomorphic and of degree d (see Fig.2.8). Indeed $\rho_0 : A'_1 \to A_0$ is a degree d covering by definition and $\pi_0 : A_0 \to B_1$ is an isomorphism because it is a restriction of an isomorphism. Define $X'_1 = X_1 \setminus \pi_0(A'_0 \setminus A_0)$, then $B_1 \subset X'_1$. Let $\rho_1 : X_2 \to X'_1$ be a degree d covering map for some Riemann surface X_2, and call $B'_2 = \rho_1^{-1}(B_1)$. Define $\pi_1 : A'_1 \to B'_2$ as a lift of f_1 to ρ_1. Then π_1 is an isomorphism, since $f_1 : A'_1 \to B_1$ is a degree d covering and $\rho_1 : B'_2 \to B_1$ is a degree d covering as well. Define $A_1 = A'_1 \cap X'_1$, and $B_2 = \pi_1(A_1)$.

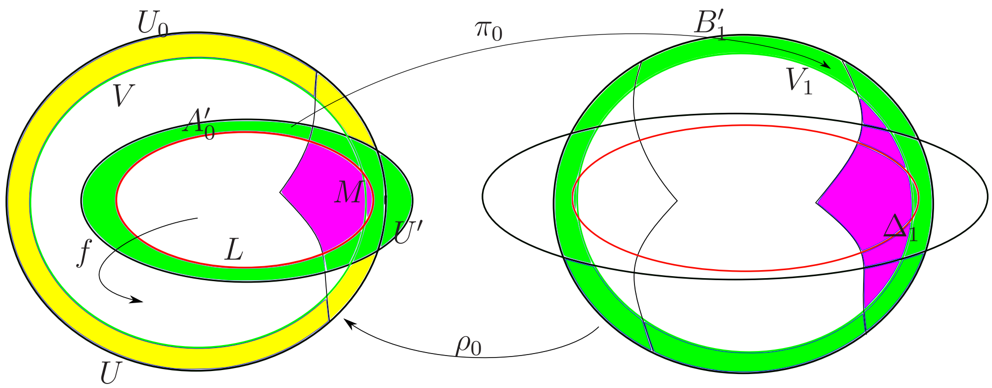


Figure 2.7: On the left: in yellow $U_0 = U \setminus \overline{V}$, in green plus purple $A'_0 = U' \setminus L$. On the right: in green plus purple $B'_1 = X''_1 \cup \Delta_1$. The map $\pi_0 : A'_0 \to B'_1$ is an isomorphism.

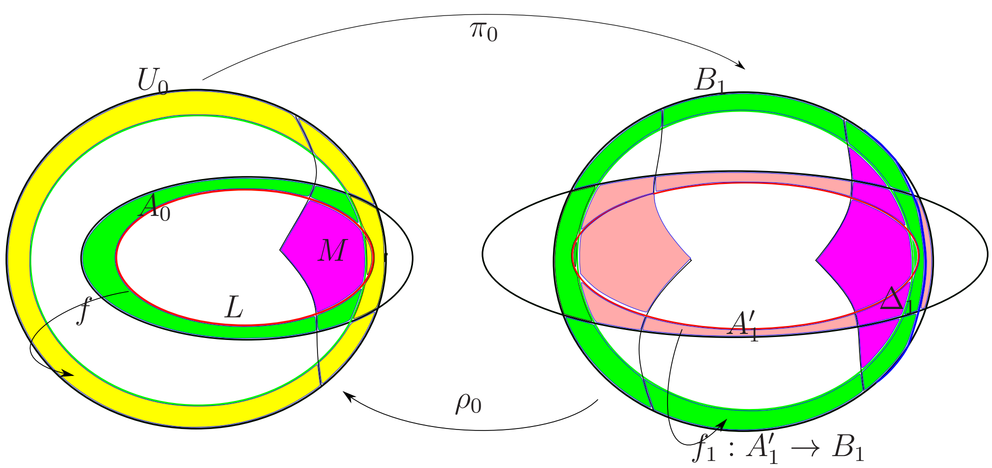


Figure 2.8: The map $f_1 = \pi_0 \circ \rho_0 : A'_1 \to B_1$ is proper holomorphic of degree d.

Define $A'_2 = \rho_1^{-1}(A_1)$ and $f_2 = \pi_1 \circ \rho_1 : A'_2 \to B_2$. The map f_2 is proper, holomorphic and of degree d, indeed $\rho_1 : A'_2 \to A_1$ is a degree d covering and $\pi_1 : A_1 \to B_2$ is an isomorphism. Define $X'_2 = X_2 \setminus \pi_1(A'_1 \setminus A_1)$, then $B_2 \subset X'_2$.

Define recursively $\rho_{n-1} : X_n \to X'_{n-1}$ for $n > 1$ as a holomorphic degree d covering for some Riemann surface X_n and call $B'_n = \rho_{n-1}^{-1}(B_{n-1})$. Define recursively $\pi_{n-1} : A'_{n-1} \to B'_n \subset X_n$ as a lift of f_{n-1} to ρ_{n-1}. Then π_{n-1} is an isomorphism. Define $A_{n-1} = A'_{n-1} \cap X'_{n-1}$, and $B_n = \pi_{n-1}(A_{n-1})$. Define $A'_n = \rho_{n-1}^{-1}(A_{n-1})$ and $f_n = \pi_{n-1} \circ \rho_{n-1} : A'_n \to B_n$. Then all the f_n are proper holomorphic maps of degree d, indeed $\rho_{n-1} : A'_n \to A_{n-1}$ are degree d coverings and $\pi_{n-1} : A_{n-1} \to B_n$ are isomorphisms. Define

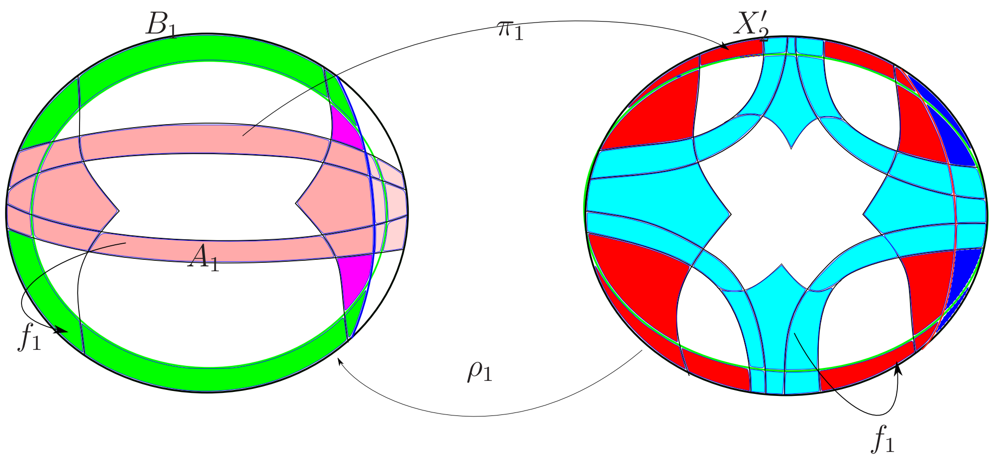


Figure 2.9: The map $\pi_1 : A_1' \to B_2'$ is a lift of f_1 to ρ_1, and it is an isomorphism.

$X_n' = X_n \setminus \pi_{n-1}(A_{n-1}' \setminus A_{n-1})$, then $B_n \subset X_n'$.

We define $X' = \coprod_{n\geq 0} X_n'$ and $X = \coprod_{n\geq 1} X_n$ (disjoint union). Let T' be the quotient of X' by the equivalence relation identifying $x \in A_n'$ with $x' = \pi_n(x) \in X_{n+1}$, and T be the quotient of X by the same equivalence relation. Then T' is an annulus, since it is constructed by identifying at each level an inner annulus $A_i \subset X_i'$ with an outer annulus $B_{i+1} \subset X_{i+1}'$ in the next level. Similarly T is an annulus, since it is constructed by identifying at each level an inner annulus $A_i' \subset X_i$ with an outer annulus $B_{i+1}' \subset X_{i+1}$ in the next level. Hence (since $\forall i > 1$, $X_i' \subset X_i$) $T \cup T' = T \cup X_0' / \sim$ is an annulus, since X_0' is an annulus and π_0 identifies an inner annulus of X_0' (which is A_0') with an outer annulus of X_1 (which is B_1'), and T is an annulus. The covering maps ρ_n induce a degree d holomorphic covering map $F : T \to T'$. Indeed, F is well defined, since at each level $f_n = \pi_{n-1} \circ \rho_{n-1}$ by definition and π_n is as a lift of f_n to ρ_n. Therefore $\rho_n \circ \pi_n = f_n = \pi_{n-1} \circ \rho_{n-1}$, and the following diagram commutes

$$\begin{array}{ccc} A_n' & \xrightarrow{\pi_n} & B_{n+1}' \\ \big\downarrow{\scriptstyle \rho_{n-1}} & & \big\downarrow{\scriptstyle \rho_n} \\ A_n & \xrightarrow{\pi_{n-1}} & B_n \end{array} \tag{2.1}$$

Finally, the map F is proper of degree d since by definition $F_{|X_n} = \rho_{n-1} : X_n \to X_{n-1}'$ is a proper map (and $F_{|X_1} = \rho_0 : X_1 \to X_0'$ is proper onto its range, which is X_0).

Now, let us construct an external map for f. Let $m > 0$ be the modulus of the annulus $T \cup T'$. Let $A \subseteq \mathbb{C}$ be any annulus with inner boundary $\mathbb{S}^1$

and modulus m. Then there exists an isomorphism

$$\alpha : T \cup T' \longrightarrow A$$

with $|\alpha(z)| \to 1$ when $z \to L$ and $\alpha(z) \to 1$ when $z \to z_0$ within $\Delta/\sim$ (where $\Delta/\sim = \{z \,|\, \exists\, n\, :\, \pi_0^{-1} \circ \ldots \circ \pi_{n-1}^{-1} \circ \pi_n^{-1}(z) \in \Delta \cup \Delta'\}$). Then we just have to repeat the construction done for the case K_f connected.

2.3.3 Properties of external maps

Let (f, U', U, γ) be a parabolic-like map of degree d, and let h_f be a representative of its external class. Then the map $h_f : \mathbb{S}^1 \to \mathbb{S}^1$ is *real analytic*, since it is the restriction to $\mathbb{S}^1$ of a holomorphic map.

The map h_f is by construction symmetric with respect to the unit circle, has a parabolic fixed point z_1 of multiplier 1 and even parabolic multiplicity $2n$, where n is the number of petals of z_0 outside K_f (where z_0 is the parabolic fixed point of f).

Let us define dividing arcs for h_f. We set $\gamma_{h_f+} := \alpha(\gamma_+ \setminus \{z_0\}) \cup \{z_1\}$, $\gamma_{h_f-} := \alpha(\gamma_- \setminus \{z_0\}) \cup \{z_1\}$ and $\gamma_{h_f} := \gamma_{h_f+} \cup \gamma_{h_f-}$ (where α is as in 2.3.1 if K_f is connected, as in 2.3.2 if not, up to real-analytic diffeomorphism). The arc γ_{h_f} divides $W'_f \setminus \mathbb{D}$, $W_f \setminus \mathbb{D}$ into Ω'_W, Δ'_W and Ω_W, Δ_W respectively, such that $h_f : \Delta'_W \to \Delta_W$ is an isomorphism and Δ'_W contains at least one attracting fixed petal of z_1 (but here Ω'_W is just contained into Ω_W).

The map α is by construction an external conjugacy between f and h_f, which extends to a topological conjugacy between f and h_f on the dividing arc γ. Hence the dividing arc γ_{h_f} inherits via α (almost all) the properties of the dividing arc γ. Indeed, since the arcs $\gamma_\pm$ are forward invariant under f, the arcs $\gamma_{h_f\pm}$ are forward invariant under h_f, and since the arcs $\gamma_\pm$ belong to repelling petals for z_0, $\gamma_{h_f\pm}$ belong to repelling petals for z_1.

Lemma 2.3.1. *The petals Ξ_+ and Ξ_- containing $\gamma_{h_f+} \setminus z_1$ and $\gamma_{h_f-} \setminus z_1$ respectively can be taken symmetric with respect to $\mathbb{S}^1$.*

Proof. Let $r(z) = 1/\bar{z}$ denote the reflection with respect to the unit circle. If the Lemma does not hold, then $r(\Xi_+) \cap \Xi_+ = \emptyset$. But then there exists at least one attracting petal Ξ in the sector bounded by γ_{h_f+} and $r(\gamma_{h_f+})$. Set $\hat{\Omega}_W = \Omega_W \cup \mathbb{S}^1 \cup r(\Omega_W)$, and let $\hat{\Omega}'_W$ be the connected component of $h_f^{-1}(\hat{\Omega}_W) \subset \hat{\Omega}_W$ having $\gamma_{h_f+}(0, 1/d) \cup r(\gamma_{h_f+}(0, 1/d))$ on the boundary. Then $h_f : \hat{\Omega}'_W \to \hat{\Omega}_W$ is an isomorphism with inverse $g_+ : \hat{\Omega}_W \to \hat{\Omega}'_W$. Note that, since $\hat{\Omega}'_W \subset \hat{\Omega}_W$, the map g_+ is a contraction for the hyperbolic metric. Choose a point $z_+ \in \hat{\Omega}_W \cap \Xi_+$ and a rectifiable path $\delta_0 \subset \hat{\Omega}_W$ from z_+ to $r(z_+)$. Define $z_n = g_+^n(z_+)$, and $\delta_n = g_+^n(\delta_0)$. Then for all $n \geq 0$,

$\delta_n \subset \hat{\Omega}'_W \subset \hat{\Omega}_W$ connects z_n to $r(z_n)$ and has hyperbolic length bounded by the hyperbolic length of δ_0. Since $z_n \to z_1$ as $n \to \infty$, $z_1 \in \partial\hat{\Omega}'_W$, and for all $n \geq 0$ the hyperbolic length of δ_n is bounded, the euclidian length of δ_n tends to zero as $n \to \infty$. But the attracting petal Ξ emerging from z_1 is repelling for g_+, and it separates z_n from $r(z_n)$, hence the euclidian length of δ_n cannot tend to zero as $n \to \infty$. Hence the repelling petal where γ_{h_f+} resides intersects the unit circle, and the same argument shows that the repelling petal where γ_{h_f-} resides intersects the unit circle. $\square$

Since there is at least one attracting fixed petal of z_1 in Δ_W, which separates the arcs γ_{h_f+} and γ_{h_f-} by an angle greater then zero, the dividing arcs of an external map cannot form a cusp.

Proposition 2.3.1. *The dividing arcs γ_{h_f+}, γ_{h_f-} are tangent to $\mathbb{S}^1$ at z_1.*

Proof. The arcs $\gamma_{h_f\pm}$ reside in repelling petals $\Xi_\pm$ of z_1. Let $\phi_\pm : \Xi_\pm \to \mathbb{H}_-$ be Fatou coordinates with axis tangent to the unit circle at the parabolic fixed point. Then $\phi_+(\mathbb{S}^1)$ is a straight line. Since γ_{h_f+} is forward invariant under h_f and $\phi_+ \circ h_f(z) = 1 + \phi_+(z)$, the curve $\phi_+(\gamma_{h_f+})$ is invariant under the map $T(z) = z+1$. This implies that the curve $\phi_+(\gamma_{h_f+})$ is 1-periodic and bounded from both above and below, and in particular (since γ_{h_f+} do not intersect the unit circle) it resides below the line $\phi_+(\mathbb{S}^1)$. Hence $\phi_+(\gamma_{h_f+})$ is tangent at infinity to $\phi_+(\mathbb{S}^1)$, and therefore the angle between them is zero. Since $\phi(z) = \Phi \circ I_n(z) \approx I_n(z) = -\frac{1}{2nz^{2n}}$, $\phi_+^{-1} \approx (-\frac{1}{2nz^{2n}})^{-1}$, the angle between γ_{h_f+} and $\mathbb{S}^1$ at z_1 is approximately $1/(2n)$ of the angle between $\phi_+(\gamma_{h_f+})$ and $\phi_+(\mathbb{S}^1)$ at infinity (which is zero). Therefore the angle between γ_+ and $\mathbb{S}^1$ at z_1 is zero, hence γ_{h_f+} is tangent to $\mathbb{S}^1$ at z_1.

On the other hand, repeating the argument above we obtain that $\phi_-(\gamma_{h_f-})$ is disjoint from $\phi_-(\mathbb{S}^1)$, hence $\phi_-(\gamma_{h_f-})$ is tangent at infinity to $\phi_-(\mathbb{S}^1)$, and therefore the arc γ_{h_f-} is tangent to $\mathbb{S}^1$ at z_1.

$\square$

An external map h_f contructed from a parabolic-like map f of degree d is an orientation preserving real-analytic map $h_f : \mathbb{S}^1 \to \mathbb{S}^1$ of the same degree d with a parabolic fixed point $z = z_1$. As we saw above, the repelling petals of z_1 intersect the unit circle, therefore in a neighborhood of the parabolic fixed point the map is expanding.

Proposition 2.3.2. *Let (f, U', U, γ) be a parabolic-like map of degree d, and let $h_f : \mathbb{S}^1 \to \mathbb{S}^1$ be a representative of its external class. Then there exists a neighborhood I of the parabolic fixed point z_1 of h_f such that*

$$|h'_f(z)| > 1, \ \ \forall z \in I \setminus \{z_1\} \ \textit{and} \ \ h'_f(z_1) = 1.$$

Proof. We can assume the parabolic fixed point at $z_1 = 1$. Set $E(x) = e^{2\pi i x}$. Lifting to $E(x)$ we obtain a map $H = E^{-1} \circ h_f \circ E : \mathbb{R} \to \mathbb{R}$ with $H(0) = 0$, $H'(0) = 1$, $H(x+1) = d + H(x)$. A neighborhood I of the parabolic fixed point is then lifted to a neighborhood $(-\epsilon, \epsilon)$ of 0. There we have:

$$H(x) = x(1 + cx^\alpha + o(x^\alpha))$$

where $\alpha = 2n > 1$ is the parabolic multiplicity of the parabolic fixed point and with $c \in \mathbb{R}_+$. Indeed c is real because H is real, and positive since the interval $(-\epsilon, \epsilon)$ resides in repelling petals. Hence $H'(x) = 1 + c(\alpha+1)x^\alpha + o(x^\alpha) > 1$ for all $x \neq 0$ in $(-\epsilon, \epsilon)$, and $H'(0) = 1$. Since $H = E^{-1} \circ h_f \circ E$, by the chain rule

$$|h_f'(z)| = |E'_{|H \circ E^{-1}(z)}| \cdot |H'_{|E^{-1}(z)}| \cdot |\frac{1}{E'}_{|E^{-1}(z)}|,$$

hence on $\mathbb{S}^1$

$$|h_f'(z)| = 2\pi \cdot |H'_{|E^{-1}(z)}| \cdot \frac{1}{2\pi} = |H'_{|E^{-1}(z)}|.$$

In particular, $|h_f'(z)| = |H'_{|E^{-1}(z)}| > 1$, $\forall z \in I \setminus \{1\}$ and $h_f'(1) = H'(0) = 1$. $\square$

Theorem 2.3.3. *Let $[h_f]$ be the external class of a parabolic-like map of degree d. Then $[h_f]$ contains a representative $h : \mathbb{S}^1 \to \mathbb{S}^1$ with $|h'(z)| > 1$ for $z \neq 1$.*

This theorem is a direct consequence of Theorem 2.3.6, integrated with Prop.2.3.4 and 2.3.5. The proof of Theorem 2.3.6 is due to Shen. We will start by proving Prop. 2.3.4 and 2.3.5, then we will include the proof of Theorem 2.3.6 for completeness, since it is not yet published.

Proposition 2.3.4. *Let (f, U', U, γ) be a parabolic-like map of degree d, and let h_f be a representative of its external class. Let $I = (-\delta_0, \delta_0)$ be a neighborhood of z_1 in $\mathbb{S}^1$. Then:*

- *$\exists K_0 > 0$ such that, for every $k \geq 0$ and $z \in \mathbb{S}^1 \setminus I$, if $\forall n \leq k$, $h_f^n(z) \notin I$, then*
$$|(h_f^k)'(z)| \geq K_0;$$
- *for every K_1 there exists n_0 such that, if $\forall n \leq n_0$, $h_f^n(z) \notin I$, then*
$$|(h_f^{n_0})'(z)| \geq K_1.$$

Proof. Set $\hat{\Omega}_W = \Omega_W \cup \mathbb{S}^1 \cup r(\Omega_W)$, and $\widehat{\Omega'}_W = h_f^{-1}(\hat{\Omega}_W)$, then $\widehat{\Omega'}_W \subset \hat{\Omega}_W$. Call ρ the coefficient of the hyperbolic metric on $\widehat{\Omega}_W$, and ρ' the coefficient of the hyperbolic metric on $\widehat{\Omega'}_W$. Since $h_f : \widehat{\Omega'}_W \to \widehat{\Omega}_W$ is a covering map, then

$$\rho'(z) = \rho(h_f(z))|h'_f(z)|,$$

and since $\widehat{\Omega'}_W \subset \widehat{\Omega}_W$, then

$$\rho'(z) > \rho(z).$$

Hence

$$||Dh_f||_\rho := \frac{\rho(h_f(z))|h'_f(z)|}{\rho(z)} > \frac{\rho(h_f(z))|h'_f(z)|}{\rho'(z)} = 1.$$

Therefore

$$||Dh_f(z)||_\rho > 1, \ \forall\, z \in \widehat{\Omega}'_W.$$

Let $I = (-\delta_0, \delta_0)$ be a neighborhood of z_1 in $\mathbb{S}^1$, then $\mathbb{S}^1 \setminus (I \cup h_f^{-1}(I))$ is a compact subset of $\widehat{\Omega}'_W$. Therefore

$$\exists\, K > 1 \mid \forall\, z \in \mathbb{S}^1 \setminus (I \cup h_f^{-1}(I)), \ \ \rho'(z) \geq K\rho(z),$$

which implies

$$||Dh_f(z)||_\rho \geq K > 1, \ \forall\, z \in \mathbb{S}^1 \setminus (I \cup h_f^{-1}(I)).$$

Since $\mathbb{S}^1 \setminus (I \cup h_f^{-1}(I))$ is a compact set, on $\mathbb{S}^1 \setminus (I \cup h_f^{-1}(I))$ the function ρ has a maximum and a minimum. Set

$$min = \min_{z \in \mathbb{S}^1 \setminus (I \cup h_f^{-1}(I))} \rho(z), \ max = \max_{z \in \mathbb{S}^1 \setminus (I \cup h_f^{-1}(I))} \rho(z),$$

then $\eta = \frac{min}{max} > 0$ because ρ is continuous and positive. Thus for all $z \in \mathbb{S}^1 \setminus (I \cup h_f^{-1}(I))$:

$$|h'_f(z)| = \frac{||Dh_f(z)||_\rho \rho(z)}{\rho(h_f(z))} \geq K\eta = K_0.$$

Given z, let $k \geq 1$ be such that, for every $n \leq k, \ h^n(z) \notin I$. Since $K > 1$, $K^k > K^{k-1}$, hence

$$|(h_f^k)'(z)| = \frac{||Dh_f^k(z)||_\rho \rho(z)}{\rho(h_f^k(z))} \geq K^k\eta \geq K_0.$$

Finally, for every K_1, choose n_0 with $K^{n_0}\eta \geq K_1$. Thus, if $\forall n \leq n_0, \ h_f^n(z) \notin I$, then

$$|(h_f^{n_0})'(z)| \geq K_1.$$

$\square$

We define an open interval $A \in \mathbb{S}^1$ to be *nice* if $h_f^n(\partial A) \cap A = \emptyset$ for all $n \geq 0$.

Proposition 2.3.5. *Let (f, U', U, γ) be a parabolic-like map of degree d, and let h_f be a representative of its external map. Then there exists an aritrary small nice interval A such that, calling z_1 the parabolic fixed point of h_f, $z_1 \in A$.*

Proof. As in the proof of the previous Lemma, set $\hat{\Omega}_W = \Omega_W \cup \mathbb{S}^1 \cup r(\Omega_W)$, and $\widehat{\Omega'}_W = h_f^{-1}(\hat{\Omega}_W)$, then $\widehat{\Omega'}_W \subset \hat{\Omega}_W$.

Call $g_i : \widehat{\Omega}_W \to G_i,\ i = 1, ..., d$ the d inverse branches of h_f. By Prop. 2.3.4, $||Dh_f(z)||_\rho > 1,\ \forall\, z \in \widehat{\Omega}'_W$, therefore $||Dg_i(z)||_\rho < 1,\ \forall\, z \in \widehat{\Omega}_W,\ \ i = 1, ..., d$. This means $d_\rho(g_i(w), g_i(z)) < d_\rho(w, z),\ \forall\, z \in \widehat{\Omega}_W,\ i = 1, ..., d$, and in particular:

$$\forall w, z \in \mathbb{S}^1 \setminus \{z_1\},\ d_\rho(g_i(w), g_i(z)) < d_\rho(w, z),\ i = 1, ..., d.$$

Iterating we obtain

$$\forall w, z \in \mathbb{S}^1 \setminus \{z_1\},\ d_\rho(g_i^n(w), g_i^n(z)) < d_\rho(w, z),\ i = 1, ..., d.$$

On the other hand, let $I = (-\delta_0, \delta_0)$ be a neighborhood of z_1 in $\mathbb{S}^1$ where $|h'_f| \geq 1$ (see Prop 2.3.2), and let $\widetilde{z} \in I$. Since the repelling petals of z_1 intersect the unit circle,

$$g_i^n(\widetilde{z}) \overset{n \to \infty}{\longrightarrow} z_1,\ i = 1, d.$$

Let $w \in \mathbb{S}^1$. Since $d_\rho(g_i^n(w), g_i^n(\widetilde{z})),\ i = 1, d$ is bounded while $g_i^n(\widetilde{z})$ tends to a boundary point when n tends to infinity:

$$g_i^n(w) \overset{n \to \infty}{\longrightarrow} z_1,\ i = 1, d.$$

Let us prove now that there exists an interval A' such that $z_1 \in A'$ and $h_f^n(\partial A') \cap A' = \emptyset$ for all $n \geq 0$. Then we will define A to be the connected component of the M-th preimage of A' containing z_1 (where M is such that $h_f^{-M}(A')$ is small as we wish).

Let us assume first $d > 2$. Then $G_2,\ G_{d-1}$ are compactly contained in $\widehat{\Omega}_W$, and $g_i : \widehat{\Omega}_W \to G_i,\ i = 2, ..., d-1$ is a strong contraction. Therefore every g_i has in $G_i \cap \mathbb{S}^1,\ i = 2, ..., d-1$ a fixed point (the fixed point belong to the unit circle because h_f is symmetric with respect to $\mathbb{S}^1$). Choose $2 \leq k \leq d-1$, then g_k has a fixed point z_k in $G_k \cap \mathbb{S}^1$. Define

$$z^1 = g_1(z_k),\ \ z^d = g_d(z_k),\ \ A' = (z^d, z^1).$$

Then $(g_d^n(z^d), g_1^n(z^1)) \subset (g_d^{n-1}(z^d), g_1^{n-1}(z^1))$, and we can choose $M > 0$ such that $A = (g_d^M(z^d), g_1^M(z^1))$ is as small as we wish.

If $d = 2$ we just repeat the construction for h_f^2.

$\square$

Theorem 2.3.6. ***Shen*** *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a topologically expanding, real analytic covering map of degree at least 2. Assume f has a parabolic fixed point p and all other periodic points of f are hyperbolic repelling. Then f is conjugate by a real analitical diffeomorphism to a metrically expanding map $g : \mathbb{S}^1 \to \mathbb{S}^1$, i.e. $|g'(z)| > 1$ for all $z \in \mathbb{S}^1$ except the unique parabolic fixed point.*

We include the proof of Shen's theorem for completeness.

Definition 2.3.7. We define $E(x) := e^{2\pi i x}$.

Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be as in the statement of Theorem 2.3.6, and let us assume the parabolic fixed point is at $z = 1$. Lifting to $E(x)$ we obtain a map $H : \mathbb{R} \to \mathbb{R}$ with $H(0) = 0,\ H'(0) = 1,\ H(x+1) = d + H(x)$. This induces a map

$$H/\sim : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$$

$$[x] \to [H(x)],$$

which we write

$$H : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$$

$$x \to H(x)$$

to simplify the notation. On a neighborhood of the parabolic fixed point the map H takes the form $H(x) = x + x^{1+\alpha} + o(x^\alpha)$, where $\alpha = 2n$ is the parabolic multiplicity of the parabolic fixed point 0.

Since

$$f = E \circ H \circ E^{-1},$$

by the chain rule

$$f' = (E \circ H \circ E^{-1})' = |E'_{|H \circ E^{-1}(z)}| \cdot |H'_{|E^{-1}(z)}| \cdot |\frac{1}{E'}_{|E^{-1}(z)}|,$$

$$\text{on } \mathbb{S}^1, \quad |f'(z)| = 2\pi \cdot |H'_{|E^{-1}(z)}| \cdot \frac{1}{2\pi} = |H'_{|E^{-1}(z)}|.$$

Therefore, in order to prove that $|f'(z)| > 1$, $\forall\, 1 \neq z \in \mathbb{S}^1$ and $f'(1) = 1$ it suffices to prove that $|H'(x)| > 1$, $\forall\, 0 \neq x \in \mathbb{R}$ and $H'(0) = 1$.

Remark 2.3.2. *In the statement of Shen's Theorem the map $f : \mathbb{S}^1 \to \mathbb{S}^1$ is topologically expanding. This hypothesis is used to ensure the existence of arbitrary small nice intervals A containing the parabolic fixed point. In our setting this properties is ensured by Prop. 2.3.5.*

The proof of the theorem is based on the following seven Lemmas.

Lemma 2.3.2. *Let $H : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a real analytic map of degree $d \geq 2$ with a parabolic fixed point at $x = 0$ with multiplier 1 and parabolic multiplicity α. Then there exist constants $\delta_0 > 0$ and $C > 0$ such that, for each $\delta \in (0, \delta_0)$, the following holds: for any $x \in (-\delta, \delta)$ with $x \neq 0$, if n is the minimal natural number such that $H^n(x) \notin (-\delta_0, \delta_0)$, then*

$$|(H^n)'(x)| \geq \frac{C}{\delta^\alpha}.$$

Proof. In a neighborhood $I \supset (-\delta_0, \delta_0)$ of the parabolic fixed point we can write $H(x) = x + x^{1+\alpha} + o(x^\alpha)$. Let $\phi_i(x) : \Xi_i \to \mathbb{C}$, $i = 1, 2$ be Fatou coordinates for the parabolic-fixed point, where Ξ_1 is a repelling petal containing the interval $(0, \delta)$ and Ξ_2 is a repelling petal containing the interval $(-\delta, 0)$, then $\phi_i \circ H \circ \phi_i^{-1}(x) = T(x) = x + 1$, $i = 1, 2$. We can write $\phi_i(x) = \Phi_i \circ I(x)$, $i = 1, 2$, where the map $I_\alpha(x) = -\frac{1}{\alpha x^\alpha}$ conjugates the map H to the map $h^*(x) = x + 1 + o(1)$ and, as Shishikura proved in [Sh], $\Phi_i' = 1 + o(1)$, $i = 1, 2$. Thus the following diagram commutes:

$$\begin{array}{ccc} \Xi_i & \xrightarrow{H} & \Xi_i \\ \downarrow{\scriptstyle I_\alpha} & & \downarrow{\scriptstyle I_\alpha} \\ \mathbb{H}_- & \xrightarrow{h^*} & \mathbb{H}_- \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} \\ \mathbb{H}_- & \xrightarrow{T} & \mathbb{H}_- \end{array} \tag{2.2}$$

Hence on both Ξ_1, Ξ_2 we can write $H^n(x) = (\Phi_i \circ I_\alpha)^{-1} \circ T^n \circ \Phi_i \circ I_\alpha(x)$, $i = 1, 2$, and therefore (from now we will avoid the subindices):

$$(H^n)'(x) = ((\Phi \circ I_\alpha)^{-1} \circ T^n \circ \Phi \circ I_\alpha(x))' =$$

$$= ((I_\alpha)^{-1})'|_{\Phi^{-1}(T^n(\Phi(I_\alpha(x))))} \cdot ((\Phi)^{-1})'|_{T^n(\Phi(I_\alpha(x)))} \cdot ((T)^n)'|_{\Phi(I_\alpha(x))} \cdot \Phi'|_{I_\alpha(x)} \cdot (I_\alpha(x))'$$

Since $T'(x) = 1$ and $\Phi' = 1 + o(1)$ (i.e. $\exists k > 1$ such that $\frac{1}{\sqrt{(k)}} < \Phi', (\Phi^{-1})' < \sqrt{(k)}$) we have

$$(H^n)'(x) > ((I_\alpha)^{-1})'|_{\Phi^{-1}(T^n(\Phi(I_\alpha(x))))} \cdot (I_\alpha(x))' \cdot 1/k,$$

and since $\Phi^{-1}(T^n(\Phi(I_\alpha(x)))) = I_\alpha(H^n(x))$ we obtain

$$(H^n)'(x) > \frac{1}{(I_\alpha^n)'|_{H^n(x)}} \cdot (I_\alpha(x))' \cdot 1/k = \frac{(H^n(x))^{\alpha+1}}{x^{\alpha+1}} \cdot 1/k.$$

Since $H^n(x) \notin (-\delta_0, \delta_0)$ and $x \in (-\delta, \delta)$, $|H^n(x)| > \delta_0$ and $|x| < \delta < 1$. Hence

$$(H^n)'(x) > \frac{(H^n(x))^{\alpha+1}}{kx^{\alpha+1}} > \frac{\delta_0^{\alpha+1}}{k\delta^{\alpha+1}} > \frac{C}{\delta^\alpha}.$$

□

Lemma 2.3.3. *There exists a small nice interval A with $0 \in A$ such that, for any $x \in A$, if $k \geq 1$ is the first return time of x into A, then*

$$|DH^k| \geq 1.$$

Proof. Let A_1 be the component of $H^{-1}(A)$ which contains 0. If $x \in A_1$, then $H(x) \in A$. Hence, by Prop. 2.3.2, for $x \in A_1$, $k = 1$, $|DH(x)| \geq 1$.

If $x \notin A_1$, let $k > 1$ be the first number such that $H^k(x) \in A$, and let J be the component of $H^{-k}(A)$ which contains x. Since A is a nice interval, $J \subset\subset A \setminus A_1$. Indeed, if $\partial A \cap H^{-k}(A) \neq \emptyset$, since A is open, $H^k(A) \cap \partial A \neq \emptyset$, which is a contradiction since A is a nice interval. On the other hand, if $H^{-1}(A) \cap H^{-k}(A) \neq \emptyset$, then $A \cap H^{k-1}(A) \neq \emptyset$ and, since A is open, $\partial A \cap H^{k-1}(A) \neq \emptyset$, which is a contradiction since A is a nice interval.

Since we are in a neighborhood of the parabolic fixed point, reducing A if necessary, we can suppose x belongs to a repelling petal for the parabolic fixed point. Thus on A the map H is conjugate by $I_n = -\frac{1}{2nx^{2n}}$ (where $2n$ is the multiplicity of the parabolic fixed point) to the map $h^*(w) = 1 + w + o(1)$. Set $A = [a_-, a_+]$ and $A_1 = [a'_-, a'_+]$, and $a_* = I_n(a'_+)$, $a_* + s = I_n(a_+)$. Then $a_* + s \approx a_* + 1$.

Since H is C^2 and has no critical points, $\log DH$ has bounded variation (by $C = \int_0^1 \frac{|D^2H(x)|}{DH(x)} dx$). Since A is a nice interval, the intervals J, $H(J)$, $H^2(J)$, ..., $H^{k-1}(J)$ are disjoint. Hence it follows from [MS] (see Corollary 2 at page 38) that H^k has uniformly bounded distortion on J.

More precisely, for all x_0, $x_1 \in J$, $e^{-C} < \frac{DH^k(x_0)}{DH^k(x_1)} < e^C$.

Remark 2.3.3. *Note that choosing $x \in \mathbb{R}/\mathbb{Z}$ determines the k (the first return time of x in A) and the J (which is the component of $H^{-k}(A)$ which contains x) for which the inequality holds. On the othert hand, C does not depend on k nor on J.*

Hence

$$|A| = a_+ - a_- = \int_{|J|} (DH^k)(x)dx < e^C \int_{|J|} (DH^k)(x_0)dx = e^C |DH^k|(x_0)|J|.$$

Since $|A| < e^C|DH^k|(x_0)|J|$, to prove that, for all $x \in A$, if $k \geq 1$ is the first return time of x into A, then $|DH^k| \geq 1$, it is enough to prove that $|J| << |A|$.

We have that $|J| < a_+ - a'_+$ and $|A| = a_+ - a_- > a_+$ (more precisely, $|J| < \max\{a_+ - a'_+, a_- - a'_-\}$. Let us assume $a_+ - a'_+ = \max\{a_+ - a'_+, a_- - a'_-\}$). Therefore

$$\frac{|J|}{|A|} < \frac{a_+ - a'_+}{a_+} = \frac{I_n^{-1}(a_* + s) - I_n^{-1}(a_*)}{I_n^{-1}(a_* + s)} =$$

$$= \frac{(-2n(a_* + s))^{-\frac{1}{2n}} - (-2na_*)^{-\frac{1}{2n}}}{(-2n(a_* + s))^{-\frac{1}{2n}}} =$$

$$= 1 - (1 + \frac{s}{a_*})^{\frac{1}{2n}} \approx \frac{1}{2n}\frac{s}{a_*} \overset{a_* \to \infty}{\to} 0,$$

which means that $\frac{|J|}{|A|} \to 0$ as $|A| \to 0$. Hence $|J| << |A|$, and, since $|A| < e^C|DH^k|(x_0)|J|$, $|DH^k| \geq 1$ for all $x \in A$. $\square$

Lemma 2.3.4. *There exists K such that, for each $x \in \mathbb{R}/\mathbb{Z}$ and $k \geq 1$, we have*

$$|DH^k| \geq K.$$

Proof. Take a small nice interval $A \ni 0$ so that the conclusion of Lemma 2.3.3 holds.

Suppose at first $x \in A$. If $H^k(x) \in A$, then $|DH^k(x)| \geq 1$ by Lemma 2.3.3. If $H^k(x) \notin A$, let $r(x)$ be the minimal number such that $H^{r(x)}(x) \notin A$. Then by Lemma 2.3.2, by shrinking A if necessary, $|DH^{r(x)}(x)| \geq 1$. On the other hand, by Prop. 2.3.4, $|DH^{k-r(x)}(H^{r(x)}(x))| \geq K_0$, thus $|DH^k(x)| \geq K_0$.

Suppose now $x \notin A$. Define $m = \min_{x \in \mathbb{R}/\mathbb{Z}} |DH(x)|$. Then $m > 0$ because $H : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is without critical points and $\mathbb{R}/\mathbb{Z}$ is compact. Let $s(x)$ be the minimal number such that $H^{s(x)}(x) \in A$. Since x, $H^{s(x)-1}(x) \notin A$, by Prop. 2.3.4 $|DH^{s(x)-1}(x)| > K_0$. Thus, $|DH^{s(x)}(x)| = |DH^{s(x)}(H^{s(x)-1}(x))| \cdot |DH^{s(x)-1}(x)| \geq mK_0$. Then we are back to the case $x \in A$, hence the result follows defining $K = mK_0$. $\square$

Lemma 2.3.5. *For each $x \in \mathbb{R}/\mathbb{Z}$, one of the following holds:*

1. *$H^k(x) = 0$ for some $k \geq 0$.*

2. *$|DH^k| \to \infty$ as $k \to \infty$.*

Proof. Assuming $H^k(x) \neq 0$ for all $k \geq 0$, let us prove that $|DH^k| \to \infty$ as $k \to \infty$. Let U be an arbitrary small neighborhood of 0, i.e. $U = (-\delta, \delta)$ for an arbitrary small $\delta > 0$. If $H^k(x) \notin U$ for all $k \geq 0$ and for any $\delta > 0$, then the result follows by Prop. 2.3.4. Assume now that for every $\delta > 0$, $H^k(x) \in U = (-\delta, \delta)$ for infinitely many k. By Lemma 2.3.4, $|DH^k(x)| \geq K$. On the other hand, since $H^n(x) \neq 0$ for all $n \geq 0$, there exists m such that $H^{k+m}(x) \notin (-\delta, \delta)$, therefore by Lemma 2.3.2 $|DH^m(H^k(x))| \geq C_0/\delta^\alpha$. Thus, it follows that

$$|DH^{k+m}(x)| \geq C_0 K \delta^{-\alpha},$$

which is large provided that δ is small. Since $H^k(x) \in U = (-\delta, \delta)$ for infinitely many k, let k_n be a sequence converging to infinity such that $H^{k_n}(x) \in U$ if n even and $H^{k_n}(x) \notin U$ if n odd. Therefore

$$|DH^{k_{2n+1}}| = \prod_{j=0}^{2n} |DH^{k_{j+1}-k_j}(H^{k_j}(x))| \geq (C_0 K \delta^{-\alpha})^n,$$

and clearly

$$|DH^{k_{2n+1}}| \overset{n\to\infty}{\longrightarrow} \infty.$$

By Lemma 2.3.4, for every $1 \leq j < k_{2n+1} - k_{2n-1}$,

$$|DH^{k_{2n-1}+j}| \geq (C_0 K \delta^{-\alpha})^{n-1} K.$$

Therefore

$$\liminf_{1 \leq i < k_{2(n+1)+1} - k_{2n+1}} |DH^{k_{2n+1}+i}| = (C_0 K \delta^{-\alpha})^n K >$$

$$> \liminf_{1 \leq j < k_{2n+1} - k_{2n-1}} |DH^{k_{2n-1}+j}| = (C_0 K \delta^{-\alpha})^{n-1} K,$$

hence,

$$\liminf_{k\to\infty} |DH^k| = \infty,$$

and finally

$$|DH^k| \overset{k\to\infty}{\longrightarrow} \infty.$$

□

Let us fix a small nice interval A with $0 \in A$.

Lemma 2.3.6. *There exists a real analytic metric $\rho = \rho(x)|dx|$ such that the following holds:*

1. $\rho'(0) = 0$ *and* $\rho''(x) > 0$ *for* $x \in A$;

2. *for any $x \in \mathbb{R}/\mathbb{Z} \setminus A$, let $s(x)$ denote the entry time of x into A, then*

$$|DH^{s(x)}|_\rho(x) \geq 2.$$

Proof. By Prop. 2.3.4, there exists N such that, whenever $s(x) > N$, we have

$$|DH^{s(x)}(x)| \geq 4.$$

Let

$$X = \{x \in \mathbb{R}/\mathbb{Z} \ : \ s(x) \leq N\}$$

and let

$$\rho_0 = inf\{|DH^{s(x)}(x)| \ : \ s(x) \leq N\}.$$

Then $\rho_0 > 0$ since $H : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is without critical points and $\mathbb{R}/\mathbb{Z}$ is compact.

The set X is the union of the first N levels of preimages of A disjoint from A. Since N is finite and A is nice, X and $\bar{A}$ are disjoint. Set $d = dist(X, \bar{A}) > 0$, and call $\partial A_+, \partial A_-$ the boundary points of A in clockwise order. Define $U(A) = [\partial A_- - d/3, \partial A_+ + d/3]$, $Y = [\partial A_- - 2d/3, \partial A_+ + 2d/3]$, and note that $A \subset\subset U(A) \subset\subset Y$ and $X \subset\subset \mathbb{S}^1 \setminus Y$. Define $m = \max\{|\partial A_+|, |\partial A_-|\}$, and, given $0 < \epsilon < 1$, set $a = \frac{1+\epsilon}{(m+d/3)^2}$. Define the map $\hat{\rho} : \mathbb{R}/\mathbb{Z} \to (0, 1+\epsilon]$ as follows:

$$\hat{\rho}(x) := \begin{cases} ax^2 & \text{on } U(A) \\ \rho_0/3 & \text{on } \mathbb{S}^1 \setminus Y \\ C^3\text{interpolation} & \text{on } Y \setminus A \end{cases}$$

Define the family $\rho_\sigma : \mathbb{R}/\mathbb{Z} \to (0, 1+\epsilon]$ as follows (where $g(0, \sigma^2)$ is a gaussian function with average 0 and variance σ small):

$$\rho_\sigma(x) = (\hat{\rho} * g(0, \sigma^2))(x) = \frac{1}{\sqrt{2\pi}\sigma} \int \hat{\rho}(w) \exp(-\frac{(x-w)^2}{2\sigma^2}) dw.$$

Let σ_0 be small such that, $\forall \sigma < \sigma_0$, $\rho_\sigma : \mathbb{R}/\mathbb{Z} \to (0, 1+\epsilon]$ is a real analytic function such that:

1. $\rho_\sigma|_A \geq 1$ and $\rho_\sigma|_X < \rho_0/2$;
2. $\rho''_\sigma(x) > 0$ for $x \in A$;
3. in A there exists a unique 0_σ such that $\rho'_\sigma(0_\sigma) = 0$.

Fix $\hat{\sigma}$ such that $dist(0, 0_{\hat{\sigma}}) < d/4$ (where $d = dist(X, \bar{A})$), and define the map

$$\rho(x) := \rho_{\hat{\sigma}}(x + 0_{\hat{\sigma}}).$$

Hence, the map $\rho : \mathbb{R}/\mathbb{Z} \to (0, 1+\epsilon]$ is a real analytic function such that:

1. $\rho|_A \geq 1$ and $\rho|_X < \rho_0/2$;
2. $\rho'(0) = 0$ and $\rho''(x) > 0$ for $x \in A$.

Then for $x \in X$, we have (since if $x \in X$, $|DH^{s(x)}(x)| > \rho_0$, $\rho|_A \geq 1$ and $\rho_X < \rho_0/2$):

$$|DH^{s(x)}(x)|_\rho = |DH^{s(x)}(x)| \frac{\rho(H^{s(x)}(x))}{\rho(x)} \geq \rho_0 \frac{1}{\rho_0/2} = 2$$

and if $s(x) > N$, then (since $\rho(x) < 2$ for all $x \in \mathbb{R}/\mathbb{Z}$, $\rho|_A \geq 1$ and if $s(x) > N$, $|DH^{s(x)}(x)| \geq 4$)

$$|DH^{s(x)}(x)|_\rho = |DH^{s(x)}(x)| \frac{\rho(H^{s(x)}(x))}{\rho(x)} > \frac{4}{2} = 2.$$

□

Lemma 2.3.7. *There exists $\delta_1 > 0$ such that $|DH^k(x)|_\rho \geq 1$ for all $x \in (-\delta_1, \delta_1)$ and $k \geq 0$.*

Proof. Note that, since 0 is the parabolic fixed point, $|DH^k(0)|_\rho = |DH^k(0)| \frac{\rho(H^k(0))}{\rho(0)} = |DH^k(0)| \frac{\rho(0)}{\rho(0)} = |DH^k(0)| = 1$ for all $k \geq 0$. For any $x \in (-\delta, \delta) \subset (-\delta_0, \delta_0) \subseteq A$, with $x \neq 0$, let $r(x)$ be the minimal positive integer such that $H^{r(x)}(x) \notin A$. As in Lemma 2.3.2, there exists $C_0 > 0$ such that

$$|DH^{r(x)}(x)| \geq C_0/\delta^\alpha.$$

Choose $\delta_1 > 0$ such that $|\delta_1| < |\delta_0|$ and

$$\delta_1^\alpha < C_0 K \eta,$$

where

$$\eta = \frac{\min_{x \in \mathbb{R}/\mathbb{Z}} \rho(x)}{\max_{x \in \mathbb{R}/\mathbb{Z}} \rho(x)}.$$

Then for $x \in (-\delta_1, \delta_1)$ and $k \geq r(x)$, we have (by Lemma 2.3.4 $|DH^{k-r(x)}(H^{r(x)})| > K,\ \forall k \geq 1$)

$$|DH^k(x)| > K|DH^{r(x)}(x)| \geq C_0 K/\delta_1^\alpha.$$

Thus

$$|DH^k(x)|_\rho = |DH^k(x)| \frac{\rho(H^k(x))}{\rho(x)} \geq |DH^k(x)|\eta \geq \frac{C_0 K \eta}{\delta_1^\alpha} > 1.$$

For $k < r(x)$, $H^k(x) \in A$. Hence we are in the repelling petal, and by Prop. 2.3.2, by shrinking A if necessary, we obtain:

$$|DH^k(x)| > 1.$$

Since $\rho'(0) = 0$ and $\rho'' > 0$ on A, we have $\rho(H^k(x)) > \rho(x)$. Hence, for any $x \in (-\delta_1, \delta_1)$,

$$|DH^k(x)|_\rho \geq |DH^k(x)| > 1.$$

□

Lemma 2.3.8. *For each $x \in \mathbb{R}/\mathbb{Z}$, there exists a neighborhood $U(x)$ and an integer $k_0 = k_0(x)$, such that:*

$$|DH^k(w)|_\rho > 1$$

for all $w \in U(x) \setminus \{0\}$ and $k \geq k_0(x)$.

Proof. By Lemma 2.3.7, the statement holds for $x = 0$. By Lemma 2.3.6, for any $x \in \mathbb{R}/\mathbb{Z} \setminus A$, if k is the minimal positive integer such that $H^k(x) \in A$, then $|DH^k(x)|_\rho \geq 2$. Hence $|DH^k(x)|_\rho > 1$ for all $0 \neq x \in \bigcup_{k=0}^{\infty} H^{-k}(U(0))$.

Hence it suffices to prove that for each x such that $H^n(x) \neq 0$ for all $n \geq 0$, there exists a neighborhood $U(x)$ and an integer $k_0 = k_0(x)$, such that for all $k \geq k_0(x)$ and for all $w \in U(x)$, we have $|DH^k(w)|_\rho > 1$.

Assume $H^n(x) \neq 0$ for all $n \geq 0$. Then, by Lemma 2.3.5, $|DH^k(x)| \to \infty$ as $k \to \infty$. Hence, by continuity, there exists a k_0 and a neighborhood $U(x)$ of x such that, for $w \in U(x)$, $|DH^{k_0}(w)|$ is big, and in particular:

$$|DH^{k_0}(w)| \geq \frac{2}{K\eta}, \ \forall w \in U(x),$$

where η is as above. By Lemma 2.3.4, for all $k \geq k_0$ and $w \in U(x)$, we have

$$|DH^k(w)| \geq \frac{2}{\eta},$$

hence

$$|DH^k(w)|_\rho \geq |DH^k(w)|\eta \geq 2.$$

□

Let us now prove Shen's Theorem.

Proof. For each $x \in \mathbb{R}/\mathbb{Z}$, let $U(x)$, $k_0(x)$ be given by Lemma 2.3.8. By compactness, there exists a finite set x_1, x_2, ..., x_n such that $\mathbb{R}/\mathbb{Z} = U(x_1) \cup U(x_2) \cup ... \cup U(x_n)$. Let $k = \max_{i=1}^{n} k_0(x_i)$. Then for any $x \in \mathbb{R}/\mathbb{Z} \setminus \{0\}$, $|DH^k(x)|_\rho > 1$.

Finally, define a metric $\widetilde{\rho}$ as

$$\widetilde{\rho} = \sum_{j=0}^{k-1} (H^j)^*(\rho).$$

Then

$$|DH(x)|_{\widetilde{\rho}} = |DH(x)| \cdot \frac{\widetilde{\rho}(H(x))}{\widetilde{\rho}(x)} = |DH(x)| \cdot (\frac{\sum_{j=0}^{k-1}(H^j)^*(\rho(H(x)))}{(\sum_{j=0}^{k-1}(H^j)^*(\rho(x)))}) =$$

$$= |DH(x)| \cdot \frac{(H^0)^*(\rho(H(x))) + (H)^*(\rho(H(x))) + ... + (H^{k-1})^*(\rho(H(x)))}{(H^0)^*(\rho(x)) + (H)^*(\rho(x)) + ... + (H^{k-1})^*(\rho(x))} =$$

$$= |DH(x)| \cdot (\frac{|DH^0(H(x))|\rho(H(x)) + |DH(H(x))|\rho(H^2(x)) + ... + |DH^{k-1}(H(x))|\rho(H^k(x))}{|DH^0(x)|\rho(x) + |DH(x)|\rho(H(x)) + ... + |DH^{k-1}(x)|\rho(H^{k-1}(x))}) =$$

$$= \frac{|DH(x)|\rho(H(x)) + |DH^2(x)|\rho(H^2(x)) + ... + |DH^k(x)|\rho(H^k(x))}{\rho(x) + |DH(x)|\rho(H(x)) + ... + |DH^{k-1}(x)|\rho(H^{k-1}(x))} =$$

$$= \frac{\sum_{j=1}^{k-1} |DH^j(x)|\rho(H^j(x)) + |DH^k(x)|\rho(H^k(x))}{\sum_{j=1}^{k-1} |DH^j(x)|\rho(H^j(x)) + \rho(x)},$$

and since

$$|DH^k(x)|\frac{\rho(H^k(x))}{\rho(x)} = |DH^k(x)|_\rho \geq 1, \ \ \forall x \in \mathbb{R}/\mathbb{Z},$$

and the equality holds only at $x = 0$, we obtain

$$|DH(x)|_{\widetilde{\rho}} \geq 1, \ \ \forall x \in \mathbb{R}/\mathbb{Z},$$

and the equality holds only at $x = 0$.

Let us find now a map $F : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, real analytically conjugate to H and such that $|F'(x)| \geq 1$ for all $x \in \mathbb{R}/\mathbb{Z}$ and the equality holds only at $x = 0$. Hence we need to find a real analytic diffeomorphism $\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ fixing the origin such that $\phi'(x) = C\widetilde{\rho}(x)$, where C is a constant. Indeed, given such a map ϕ, we can define

$$F(x) := \phi \circ H \circ \phi^{-1},$$

thus F is real analytically conjugate to H and

$$|F'(x)| = |\phi'_{|H(\phi^{-1}(x))}| \cdot |H'_{\phi^{-1}(x)}| \cdot |(\phi^{-1})'_{(x)}| =$$

$$\frac{|\phi'_{|H(\phi^{-1}(x))}| \cdot |H'_{\phi^{-1}(x)}|}{|\phi'_{\phi^{-1}(x)}|} =$$

$$\frac{|C\widetilde{\rho}(H(\phi^{-1}(x))| \cdot |H'_{\phi^{-1}(x)}|}{C\widetilde{\rho}(\phi^{-1}(x))} =$$

$$|DH(\phi^{-1}(x))|_{\widetilde{\rho}} \geq 1, \ \ \forall x \in \mathbb{R}/\mathbb{Z},$$

and moreover the equality holds only at $x = 0$.

Since $\widetilde{\rho}$ is real analytic, such a ϕ is given by:

$$\phi(x) := \int_0^x C\widetilde{\rho}dx + \phi(0) = C\int_0^x \widetilde{\rho}dx,$$

where

$$\frac{1}{C} = \int_0^1 \widetilde{\rho}dx;$$

and then the theorem follows. □

2.3.4 Parabolic external maps

So far we have constructed external maps from parabolic-like maps, thus we have considered external maps only in relation to parabolic-like maps.

We now want to separate these two concepts, and then consider external maps as maps of the unit circle to itself with some specific properties, without refering to a particular parabolic-like map. In order to do so we need to give an abstract definition of external map, which endows it with all the properties it would have, if it would have been constructed from a parabolic-like map.

An external map h_f contructed from a parabolic-like map f of degree d is an orientation preserving, real-analytic and, up to conjugacy, metrically expanding (i.e. $|h_f'(z)| \geq 1, \ \forall z \in \mathbb{S}^1$) map $h_f : \mathbb{S}^1 \to \mathbb{S}^1$ of the same degree d with precisely one parabolic fixed point $z = z_1$ and all the other periodic points repelling.

Definition 2.3.8. (Singly parabolic external map) Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation preserving real-analytic and metrically expanding (i.e. $|h'(z)| \geq 1$) map of degree $d > 1$. We say that h is a *singly parabolic external map,* if h has precisely one parabolic fixed point, i.e. if there exists a unique $z = z_*$ such that $h(z_*) = z_*$, $h'(z_*) = 1$ and $|h'(z)| > 1$ for all $z \neq z_*$.

The multiplicity of z_* as parabolic fixed point of h is even and in particular greater than 1, since the map h is symmmetric with respect to the unit circle (exactly as for the fixed point of an external map constructed from a parabolic-like map). As the map h is metricly expanding, the repelling petals of z_* intersect the unit circle. Therefore we can construct dividing arcs.

Proposition 2.3.9. *Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be a singly parabolic external map of degree $d > 1$, $h : W' \to W$ an extension which is a degree d covering (where $W = \{z : e^{-\epsilon} < |z| < e^{\epsilon}\}$ for an $\epsilon > 0$, and $W' = h^{-1}(W)$) and call z_* its parabolic fixed point. Then there exist forward invariant arcs $\widetilde{\gamma}_+ : [0,1] \to W \setminus \mathbb{D}$ and $\widetilde{\gamma}_- : [0,-1] \to W \setminus \mathbb{D}$ such that $\widetilde{\gamma}_\pm(0) = z_*$ and*

$$h(\widetilde{\gamma}_\pm(t)) = \widetilde{\gamma}_\pm(dt) \ \forall - \frac{1}{d} \le t \le \frac{1}{d}.$$

The arcs $h(\widetilde{\gamma}_\pm(t))$ are tangent to $\mathbb{S}^1$ at z_*. We call $\widetilde{\gamma}_\pm$ dividing arcs.

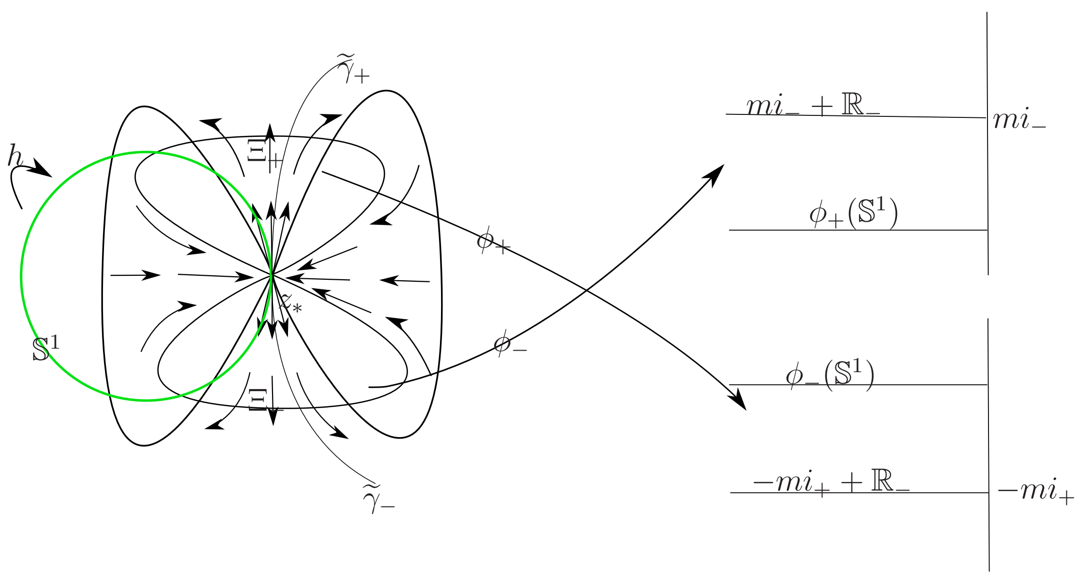


Figure 2.10: The arcs $\widetilde{\gamma}_\pm$ are preimages of horizontal lines by repelling Fatou coordenates with axis tangent to the unit circle at the parabolic fixed point z_*.

Proof. We choose the arcs $\widetilde{\gamma}_\pm$ to be preimages of horizontal lines by repelling Fatou coordenates with axis tangent to the unit circle at the parabolic fixed point z_* (see Fig. 2.10).

Since the map $h : \mathbb{S}^1 \to \mathbb{S}^1$ is expanding, the repelling petals $\Xi_\pm$ intersect the unit circle. Let $\phi_\pm : \Xi_\pm \to \mathbb{H}_-$ be Fatou coordinates with axis tangent to the unit circle at the parabolic fixed point z_*. As h is reflection symmetric with respect to $\mathbb{S}^1$, the image of the unit circle in the Fatou coordinate planes are horizontal lines, which we can suppose coincide with $\mathbb{R}_-$, possibly changing the normalizations of $\phi_\pm$. Let $z_\pm$ be intersection points of $\Xi_\pm$ respectively and the outer boundary of W. Thus $\phi_+(z_+) = -im_+,\ \phi_-(z_-) = im_-,\ \exists m_+, m_- > 0$.

Let us define

$$\widetilde{\gamma}_+ := \phi_+^{-1}(-m_+ i + \mathbb{R}_-)$$
$$\widetilde{\gamma}_- := \phi_-^{-1}(m_- i + \mathbb{R}_-).$$

Reparametrizing the arcs as

$$\widetilde{\gamma}_+(t) = \phi_+^{-1}(log_d(t) - im_+),$$

$$\widetilde{\gamma}_-(t) = \phi_-^{-1}(log_d(-t) + im_-)$$

we obtain $\widetilde{\gamma}_+ : [0,1] \to W \setminus \mathbb{D},\ \widetilde{\gamma}_- : [0,-1] \to W \setminus \mathbb{D}$ and

$$h(\widetilde{\gamma}_\pm(t)) = \widetilde{\gamma}_\pm(dt)\ \forall -\frac{1}{d} \leq t \leq \frac{1}{d}.$$

$\square$

Let $h : W' \to W$ be an extension which is a degree d covering (where $W = \{z : e^{-\epsilon} < |z| < e^{\epsilon}\}$ for an $\epsilon > 0$, and $W' = h^{-1}(W)$) of the map $h : \mathbb{S}^1 \to \mathbb{S}^1$. The dividing arcs $\widetilde{\gamma}_\pm$ constructed in Prop. 2.3.9 divide $W' \backslash \mathbb{D},\ W \backslash \mathbb{D}$ into Ω'_W, Δ'_W and Ω_W, Δ_W respectively, such that $h : \Delta'_W \to \Delta_W$ is an isomorphism and Δ'_W contains at least an attracting fixed petal of z_*, and $\Omega_W \setminus \Omega'_W$ is a *topological quadrilateral.*

Lemma 2.3.9. *Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be a singly parabolic external map of degree $d > 1$. Then there exist W', W neighborhoods of $\mathbb{S}^1$ for an extension $h : W' \to W$ such that $\Omega_W \setminus \Omega'_W$ is a topological quadrilateral.*

Proof. Let us assume the parabolic fixed point is 1. Let $\delta > 0$ be such that, defining $\hat{W}' = \{z : e^{-\delta} < |z| < e^{\delta}\}$ and $\hat{W} = h(\hat{W})$, $\partial\hat{W} \cap \mathbb{S}^1 = \emptyset$. Let $\widetilde{\gamma}_\pm$ be dividing arcs as in Prop. 2.3.9, which devide $\hat{W} \setminus \mathbb{D}$, $\hat{W}' \setminus \mathbb{D}$ in $\hat{\Omega}$, $\hat{\Delta}$ and $\hat{\Omega}'$, $\hat{\Delta}'$ respectively. By definition of singly parabolic external map, $|h'(z)| > 1$ for all $z \neq 1$ (we assume the parabolic fixed point is 1). Hence, by continuity and since the dividing arcs are tangent to $\mathbb{S}^1$ at 1, and they are forward invariant, $|h'(z)| > 1$ for all $z \in \hat{W}' \backslash (\hat{\Delta}' \cup r(\hat{\Delta}'))$ (where $r(z) = 1/\bar{z}$).

Choose $0 < \epsilon < \delta$ such that $W = \{z : e^{-\epsilon} < |z| < e^{\epsilon}\}$ is compactly contained into $\hat{W}$, and set $W' = h^{-1}(W)$. Then $h : W' \to W$ is a degree

d covering, and $|h'(z)| > 1$ for all $z \in \overline{W}' \setminus (\Delta' \cup r(\Delta'))$. Let us prove that $\Omega \setminus \Omega'$ is a topological quadrilateral.

Recall $E(z) = e^{2\pi iz}$ (see Definition 2.3.7), and let $H : S_\delta = \{z = x + iy : |y| < \delta\} \to E^{-1}(\hat{W})$ be a lift of $E \circ h$ to E, i.e., the following diagram commutes:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "H"] \arrow[d, "E"] & \mathbb{R} \arrow[d, "E"] \\ \mathbb{S}^1 \arrow[r, "h"] & \mathbb{S}^1 \end{tikzcd} \tag{2.3}$$

Then H is biholomorphic. Since

$$|H'(z)| = |(E^{-1} \circ h \circ E(z))'| = |\frac{1}{E'_{|E^{-1}\circ h \circ E(z)}}| \cdot |h'_{|E(z)}| \cdot |E'_{|z}|,$$

$$\text{on } \mathbb{R}, \quad |H'(z)| = \frac{1}{2\pi} \cdot |h'_{|E(z)}| \cdot 2\pi = |h'_{|E(z)}| > 1.$$

Hence, by continuity and since $|H'(z)| > 1$ on the preimages under the exponential map of the repelling petals of the parabolic fixed point $z = 1$, $|H'(z)| > 1$ on $S_\delta \setminus E^{-1}(\hat{\Delta}' \cup r(\hat{\Delta}'))$ (if δ is small enough). Set $S_\epsilon = E^{-1}(W)$, and $V' = E^{-1}(W')$. Then $S_\epsilon = \{z = x+iy : |y| < \epsilon < \delta\}$ is a strip compactly contained into S_δ.

Let Ω'_e be the connected component of $E^{-1}(\Omega')$ containing $]0, 1[$ in its boundary. Note that $|H'(z)| > 1$ on $E^{-1}(\overline{W}' \setminus (\Delta' \cup r(\Delta'))$, hence $|H'(z)| > 1$ on $\overline{\Omega}'_e$. Let Ω_e be the connected component of $E^{-1}(\Omega)$ containing $]0, d[$ in its boundary.

Clearly to prove that $\Omega \setminus \Omega'$ is a topological quadrilateral it suffices to prove that $\Omega_e \setminus \Omega'_e$ is a topological quadrilateral. By construction, if $z \in \partial S_\epsilon$, $Im(z) = \epsilon$, thus if $z \in \partial S_\epsilon \cap \partial\Omega_e$, $Im(z) = \epsilon$. Hence in order to prove that $\Omega \setminus \Omega'$ is a topological quadrilateral, it is enough to show that for every $z \in S_\epsilon \cap \partial\Omega_e,\ Im(H^{-1}(z)) < \epsilon$.

Let $z = x + iy \in S_\epsilon \cap \partial\Omega_e$ and write $w = H^{-1}(z)$. Let us prove that $Im(w) < \epsilon$. Let $k : [0, \epsilon] \to \mathbb{C}$ be an arc with unitary speed connecting z to the real line. Call l the length of the preimage under H of k in Ω'_e, then $Im(w) = Im(H^{-1}(z)) \leq l < \epsilon = Im(z)$. Indeed $l = \int_0^\epsilon |(H^{-1} \circ k(t))'|dt = \int_0^\epsilon |H^{-1}|'|_{(k(t))} \cdot |k(t)'|dt = \int_0^\epsilon |(H^{-1})'|_{(k(t))}dt$. Since $|H'(z)| > 1$ for all $z \in \overline{\Omega}'_e$, $\int_0^\epsilon |(H^{-1})'|(k(t))dt < \int_0^\epsilon 1dt = \epsilon$.

$\square$

Notation. • *A parabolic-like map as defined in 2.2.1 is called singly parabolic, because its external class is singly parabolic. We can generalize this concept to parabolic-like maps with external map with several* *parabolic fixed points. A general parabolic-like map has as many pairs of invariant arcs $\gamma_\pm$ (which divide U and U' in $\Omega, \Delta_1, \Delta_2, \ldots, \Delta_n$ and $\Omega', \Delta'_1, \Delta'_2, \ldots, \Delta'_n$ respectively) as the number of parabolic fixed points. We have chosen to give here the definition of singly parabolic-like map, instead of the general one, in order to simplify the notation.*

- *Also we are considering maps with a parabolic fixed point, rather than a parabolic periodic orbit, in order to simplify the notation.*

In these last two sections we saw that an external map constructed from a parabolic-like map is a singly parabolic external map, and by definition a singly parabolic external map has all the properties it would have, if it would have been constructed from a parabolic-like map. Hence in the remainder of this thesis we will not distinguish sharply between these two maps, but we will refer to both of them as *parabolic external maps*.

Definition 2.3.10. A *holomorphic degree d covering extension* $h : W' \to W$ of a parabolic external map $h : \mathbb{S}^1 \to \mathbb{S}^1$ is an extension to some neighborhood $W = \{z : e^{-\epsilon} < |z| < e^{\epsilon}\}$ for an $\epsilon > 0$, and $W' = h^{-1}(W)$ such that the map $h : W' \to W$ is a degree d covering and there exist (we can construct) dividing arcs $\widetilde{\gamma}_\pm$ which divide $W' \setminus \mathbb{D}$, $W \setminus \mathbb{D}$ into Ω'_W, Δ'_W and Ω_W, Δ_W respectively, such that $h : \Delta'_W \to \Delta_W$ is an isomorphism, Δ'_W contains at least an attracting fixed petal of the parabolic fixed point and $\Omega_W \setminus \Omega'_W$ is a topological quadrilateral.

Finally, the concept of *parabolic-like restriction* applies to parabolic external maps. Let $h : W' \to W$ be a degree d covering extension of a parabolic external map $h : \mathbb{S}^1 \to \mathbb{S}^1$. Then $h : \hat{W}' \to \hat{W}$ is a parabolic-like restriction of $h : W' \to W$ if $\hat{W} \subset W$ and $h : \hat{W}' \to \hat{W}$ is a degree d covering extension of the parabolic external map $h : \mathbb{S}^1 \to \mathbb{S}^1$.

Proposition 2.3.11. *Let $h_i : \mathbb{S}^1 \to \mathbb{S}^1$, $i = 1, 2$ be parabolic external maps of the same degree d, let $h_i : W'_i \to W_i$ be extensions which are degree d coverings, γ_i dividing arcs, and z_i their parabolic fixed points. Then:*

1. *if γ_1, γ_2 are dividing arcs as in Prop. 2.3.9, then the map $\phi_2^{-1} \circ \phi_1 : (\gamma_1) \to \gamma_2$ is a quasisymmetric conjugacy between $h_{1|\gamma_1}$ and $h_{2|\gamma_2}$;*

2. *the dividing arcs γ_i are defined up to isotopy. Hence, if γ_s is a forward invariant arc under h_1 outside $\mathbb{S}^1$, with $\gamma_s(0) = \gamma_1(0)$, γ_{+s} living in the same petal as γ_{+1} and γ_{-s} living in the same petal as γ_{-1}, then the arc γ_s is a dividing arc for h_1;*

3. *in particular, if $\gamma_s = \gamma_{s+} \cup \gamma_{s-}$, where γ_{s+} and γ_{s-} are the preimages of straight lines under Fatou coordinates for the parabolic fixed point of h_1, there exists a map $\delta : \gamma_1 \to \gamma_s$ which is a quasisymmetric conjugacy between h_1 and itself.*

Proof. Property 2 comes from the definition of equivalence for dividing arcs, and from the fact that in the domain of an extension which is a degree d covering there are no critical points (see lemma 2.2.1), we leave the details to the reader. Property 3 is a consequence of property 2, but we give the proof here anyway because of its importance in the construction of a diffeomorphic motion in chapter 3.

(1). To fix the notation let us assume the multiplicity of z_i as parabolic fixed point of h_i is $2n_i$, where $i = 1, 2$.

By an iterative local change of coordinates applied to eliminate lower order terms one by one, we obtain conformal diffeomorphisms g_i, $i = 1, 2$ which conjugate h_i to the map $z \to z(1 + z^{2n_i} + cz^{4n_i} + O(z^{6n_i}))$ on $\Xi_{i\pm}$ (where $\Xi_{i\pm}$) are the repelling petals where $\gamma_{i\pm}$ reside). Since the forward invariant arcs $\gamma_{i\pm}$ reside in the repelling petals $\Xi_{i\pm}$, it suffices to consider $h_i(z) = z(1 + z^{2n_i} + cz^{4n_i} + O(z^{6n_i}))$.

The map $I_i(z) = -\frac{1}{2n_i z^{2n_i}}$ conjugates h_i to $h_i^*(z) = z + 1 + \hat{c}_i \frac{1}{z} + O(\frac{1}{z^2})$. Shishikura proved in [Sh] that Fatou coordinates which conjugate the map h_i^* to $T(z) = z + 1$ on $I_i(\Xi_{i\pm})$ take the form $\Phi_{i\pm}(z) = z - \hat{c}_i \, log(z) + c_{i\pm} + o(1)$.

Therefore $\phi_{i\pm} = \Phi_{i\pm} \circ I_i$, and since (see Prop.2.3.9) $\gamma_{i+} = \phi_{i+}^{-1}(-m_+ i + \mathbb{R}_-)$, $\exists m_+ > 0$ and $\gamma_{i-} = \phi_{i-}^{-1}(m_- i + \mathbb{R}_-)$, $\exists m_- > 0$, we can write:

$$\gamma_{i+} = (\Phi_{i+} \circ I_i)^{-1}(-m_+ i + \mathbb{R}_-)_{\{i=1,2\}},$$
$$\gamma_{i-} = (\Phi_{i-} \circ I_i)^{-1}(m_- i + \mathbb{R}_-)_{\{i=1,2\}}.$$

$$\begin{tikzcd} \gamma_i \arrow[r, "h_i"] \arrow[d, "I_i"] & \gamma_i \arrow[d, "I_i"] \\ \mathbb{H}_- \arrow[r, "h_i^*"] \arrow[d, "\Phi_i"] & \mathbb{H}_- \arrow[d, "\Phi_i"] \\ \mathbb{H}_- \arrow[r, "T"] & \mathbb{H}_- \end{tikzcd} \tag{2.4}$$

Call $\gamma_{i+}^* = I_i(\gamma_{i+})$, $\gamma_{i-}^* = -I_i(\gamma_{i-})$ and $\gamma_i^* = \gamma_{i+}^* \cup \infty \cup \gamma_{i-}^*$, and normalize $\Phi_{i\pm}$ such that $\Phi_{i+}(\gamma_{i+}^*) = (-\infty, -1]$, and $-\Phi_{i-}(-\gamma_{i-}^*) = [1, \infty)$, $i = 1, 2$.

The map $\widehat{I_i} : \gamma_i \to \gamma_i^*$:

$$\widehat{I}_1(z) = \begin{cases} I_i(z) & \text{on } \gamma_{i+} \\ -I_i(z) & \text{on } \gamma_{i-} \end{cases}$$

is quasisymmetric on a neighborhood of 0. Define $\widehat{\mathbb{R}} = \mathbb{R} \cup \infty$, and the map $\Phi_i : \gamma_i^* \to \widehat{\mathbb{R}} \setminus]-1, 1[$ as follows:

$$\Phi_i(z) = \begin{cases} \Phi_{i+}(z) & \text{on } \gamma_{i+}^* \\ -\Phi_{i-}(-z) & \text{on } \gamma_{i-}^* \end{cases}$$

The map Φ_i is the restriction to $\gamma_i^* \setminus \infty$ of a conformal map. Again by Shishikura [Sh] the maps Φ_{i+}, Φ_{i-} have derivatives $\Phi'_{i\pm} = 1 + o(1)$, hence the map $\Phi_i : \gamma_i^* \to \widehat{\mathbb{R}} \setminus]-1, 1[$ is a diffeomorphism (one may take $1/x$ as a chart).

The map $\Phi_i \circ \widehat{I}_i$ conjugates the map h_i to the map $T_+(z) = z + 1$ on γ_{i+}, and to the map $T_-(z) = z - 1$ on γ_{i-}. Hence $\phi_2^{-1} \circ \phi_1 = (\Phi_2 \circ \widehat{I}_2)^{-1} \circ (\Phi_1 \circ \widehat{I}_1) : \gamma_1 \to \gamma_2$. The map Φ_2^{-1} is a diffeomorphism because it has the same analytic expression as Φ_2, and therefore the map $\Phi_2^{-1} \circ \Phi_1$ is a diffeomorphism. Since the maps $\widehat{I}_2$ and $\widehat{I}_1$ are quasisymmetric on a neighborhood of $z = 0$, their inverse are quasisymmetric on a neighborhood of ∞. Hence the composition $\phi_2^{-1} \circ \phi_1 = \widehat{I}_2^{-1} \circ \Phi_2^{-1} \circ \Phi_1 \circ \widehat{I}_1 : \gamma_1 \to \gamma_2$ is quasisymmetric.

(3). The proof of (3) resembles the proof of (1). Call $\gamma_{1+}^* = \phi_1(\gamma_{1+})$, $\gamma_{1-}^* = -\phi_1(\gamma_{1-})$ and $\gamma_1^* = \gamma_{1+}^* \cup \infty \cup \gamma_{1-}^*$. On the other hand, choose $m_+, m_- > 0$ and set $\gamma_{s+}(t) = \phi_{1+}^{-1}(\log_d(t) - m_+ i)$, $0 \leq t \leq 1$ and $\gamma_{s-}(t) = \phi_{1-}^{-1}(\log_d(-t) + m_- i)$, $-1 \leq t \leq 0$. Then γ_{s+} resides in the same petal as γ_{1+} and γ_{s-} resides in the same petal as γ_{1-}. Call $\gamma_{s+}^* = \phi_1(\gamma_{s+})$, $\gamma_{s-}^* = -\phi_1(\gamma_{s-})$, then $\gamma_s^* = \gamma_{s+}^* \cup \infty \cup \gamma_{s-}^*$ is the straight line passing through infinity connecting $-im_-$ and $-im_+$. Set

$$\delta_+ : \phi_{1+}(\gamma_{1+}) \to \phi_{1+}(\gamma_{s+})$$

$$\delta_+(\phi_{1+}(\gamma_{1+}(t))) = \log_d(t) - m_+ i,$$

$$\delta_- : \phi_{1-}(\gamma_{1-}) \to \phi_{1-}(\gamma_{s-})$$

$$\delta_-(\phi_{1-}(\gamma_{1-}(t))) = \log_d(-t) + m_- i,$$

and $\delta : \gamma_1^* \to \gamma_s^*$ as follows:

$$\delta(z) = \begin{cases} \delta_+(z) & \text{on } \gamma_{1+}^* \\ -\delta_-(-z) & \text{on } \gamma_{1-}^* \end{cases}$$

Define the map $S(z) = -z$, and the map $\hat{\delta} : \gamma_1 \to \gamma_s$ as follows

$$\hat{\delta}(z) = \begin{cases} \phi_{1+}^{-1} \circ \delta \circ \phi_{1+}(z) & \text{on } \gamma_{1+} \\ \phi_{1-}^{-1} \circ S \circ \delta \circ S \circ \phi_{1-}(z) & \text{on } \gamma_{1-} \end{cases}$$

The map $\hat{\delta}$ is a conjugacy between h_1 and itself, indeed $\hat{\delta} \circ h_1(\gamma_{1+}(t)) = \hat{\delta}(\gamma_{1+}(dt)) = \phi_{1+}^{-1} \circ \delta_+ \circ \phi_{1+}(\gamma_{1+}(dt)) = \phi_{1+}^{-1}(\log_d(dt) - m_+ i) = \phi_{1+}^{-1}(\log_d(t) -$

$m_+i+1) = \phi_{1+}^{-1}(\phi_{1+}(\gamma_{s+}(t))+1) = \phi_{1+}^{-1}(\phi_{1+}(h_1(\gamma_{s+}(t)))) = h_1(\gamma_{s+}(t)) = h_1 \circ \hat{\delta}(\gamma_{1+}(t))$, and similar computations shows that $\hat{\delta} \circ h_1(\gamma_{1-}(t)) = h_1 \circ \hat{\delta}(\gamma_{1-}(t))$. Therefore, since Fatou coordinates are conformal maps with quasisymmetric extension at infinity (see (1)), and the map S is conformal, in order to prove that the map $\hat{\delta}$ is a quasisymmetric conjugacy between h_1 and itself it suffices to prove that the map δ is quasisymmetric. Clearly the map δ is a diffeomorphism on $\gamma^* \setminus \infty$, hence quasisymmetric. Let us show that the map δ is a diffeomorphism in a neighborhood of infinity. Therefore, let us show that:

$$\lim_{s\to 0} \frac{1/\delta(1/s) - 0}{s - 0} = \lim_{s\to 0} \frac{1}{\delta(1/s)s} = 1.$$

Let us show first that the function $p(z) = \delta(z) - z$ is periodic, and hence bounded. Indeed $\delta(\phi_1(\gamma_1(t)) + 1) = \delta(\phi_1(h_1(\gamma_1(t)))) = \delta(\phi_1(\gamma_1(dt))) = \log_d(dt) - (\pm m_\pm)i = \log_d(t) - (\pm m_\pm)i + 1 = \delta(\phi_1(\gamma_1(t))) + 1$, hence $\delta(z+1) = \delta(z)+1$ and finally $\delta(z+1) - (z+1) = \delta(z) + 1 - z - 1 = \delta(z) - z$. Therefore $p(z) = \delta(z) - z$ is bounded. Hence

$$\lim_{s\to 0} \frac{1}{\delta(1/s)s} = \lim_{s\to 0} \frac{1}{(p(1/s) + 1/s)s} = \lim_{s\to 0} \frac{1}{(p(1/s)s + 1)} \to 1,$$

since $s \to 0$ and $p(s)$ is bounded. Hence δ is a diffeomorphism, thus it is quasisymmetric, and then the map $\hat{\delta} : \gamma_1 \to \gamma_s$ is a quasisymmetric conjugacy between h_1 and itself. $\square$

2.4 Conjugacy between parabolic-like maps

The aim of this section is to prove that, given a parabolic-like map of degree d and a parabolic external map of the same degree d, we can construct a parabolic-like map which is hybrid conjugate to the given parabolic-like map and which has as external map the given one. We start by defining notions of conjugacies between parabolic-like maps.

Remark 2.4.1. *Remember that if (f, U', U, γ) is a parabolic like map, we can consider $\gamma : [-1, 1] \to U$ as $\gamma := \gamma_+ \cup \gamma_-$, where $\gamma_+(t) = \gamma(t), t \in [0, 1]$, and $\gamma_-(t) = \gamma(-t), t \in [0, 1]$. The arcs $\gamma_\pm$ are C^1 and defined up to isotopy (see 2.2)*

Definition 2.4.1. (Conjugacy for parabolic-like mappings)
Let $(f, U', U, \gamma_{+f}, \gamma_{-f})$ and $(g, V', V, \gamma_{+g}, \gamma_{-g})$ be two parabolic-like mappings.

We say that f and g are *topologically conjugate* if there exist parabolic-like restrictions $(f, A', A, \gamma_{+f}, \gamma_{-f})$ and $(g, B', B, \gamma_{+g}, \gamma_{-g})$, and a homeomorphism $\phi : A \to B$ such that $\phi(\gamma_{\pm f}) = \gamma_{\pm g}$ and

$$\phi(f(z)) = g(\phi(z)) \quad \text{on } \Omega'_{A_f} \cup \gamma_f$$

If moreover ϕ is quasiconformal (and $\bar{\partial}\phi = 0$ a.e. on K_f), we say that f and g are *quasiconformally* (*hybrid*) conjugate.

Remark 2.4.2. *A topological (quasiconformal) conjugacy between parabolic-like maps is a (quasiconformal) homeomorphism defined on a neighborhood of the Julia set, which conjugates dynamics just on $\Omega' \cup \gamma$. This definition allows flexibility regarding the parabolic multiplicity of the parabolic fixed points (i.e. two parabolic-like maps topologically (quasiconformally) conjugate do not need to have the same number of petals).*

Definition 2.4.2. (External equivalence)
Let $(f, U', U, \gamma_{+f}, \gamma_{-f})$ and $(g, V', V, \gamma_{+g}, \gamma_{-g})$ be two parabolic-like mappings.

If K_f and K_g are connected, we say that f and g are *externally equivalent* if there exist parabolic-like restrictions $(f, A', A, \gamma_{+f}, \gamma_{-f})$ and $(g, B', B, \gamma_{+g}, \gamma_{-g})$, and a biholomorphic map

$$\psi : (A \cup A') \setminus K_f \to (B \cup B') \setminus K_g$$

such that $\psi(\gamma_{\pm f}) = \gamma_{\pm g}$ and $\psi \circ f = g \circ \psi$.

Remark 2.4.3. *Two parabolic-like maps f and g with connected filled Julia sets are externally equivalent if and only if their external maps are conjugate by a real-analytic diffeomorphism, i.e. if and only if their external maps belong to the same external class.*

In the other case (filled Julia set not connected), we say that f and g are *externally equivalent* if their external maps are conjugate by a real-analytic diffeomorphism.

Remark 2.4.4. *Note that, if ϕ is a conjugacy between two parabolic-like maps f and g, then by continuity $\phi(\gamma_f) = \gamma_g$ implies $\phi(\Omega_{A_f}) = \Omega_{B_g}$ and $\phi(\Delta_{A_f}) = \Delta_{B_g}$.*

Definition 2.4.3. (Holomorphic equivalence)
Let $(f, U', U, \gamma_{+f}, \gamma_f)$ and $(g, V', V, \gamma_{+g}, \gamma_g)$ be two parabolic-like mappings.

We say that f and g are *holomorphically equivalent* if there exist parabolic-like restrictions $(f, A', A, \gamma_{+f}, \gamma_{-f})$ and $(g, B', B, \gamma_{+g}, \gamma_{-g})$, and a biholomorphic map $\phi : (A \cup A') \to (B \cup B')$ such that $\phi(\gamma_{\pm f}) = \gamma_{\pm g}$ and

$$\phi(f(z)) = g(\phi(z)) \quad \text{on } A \cup A'$$

Lemma 2.4.1. *Let $f_i : U'_i \to U_i$, $i = 1,2$ be two parabolic-like mappings with disconnected Julia sets.*

Let $W_i \approx \mathbb{D}$ be a full relatively compact connected subset of U_i containing $\overline{\Omega'_i}$ and the critical values of f_i, and such that $f_i : f_i^{-1}(W_i) \to W_i$ is a parabolic-like restriction of $(f_i, U_i, U'_i, \gamma_i)$. Define $L_i := f_i^{-1}(\overline{W}_i) \cap \overline{\Omega}'_i$.

Suppose $\overline{\phi} : (U_1 \cup U'_1) \setminus L_1 \to (U_2 \cup U'_2) \setminus L_2$ is a biholomorphic map such that

$$\begin{array}{ccc} U'_1 \setminus L_1 & \xrightarrow{f_1} & U_1 \setminus \overline{W_1} \\ \Big\downarrow \overline{\phi} & & \Big\downarrow \overline{\phi} \\ U'_2 \setminus L_2 & \xrightarrow{f_2} & U_2 \setminus \overline{W_2} \end{array} \tag{2.5}$$

Then h_1 and h_1 are analytically conjugate, and we say that $\overline{\phi} : (U_1 \cup U'_1) \setminus L_1 \to (U_2 \cup U'_2) \setminus L_2$ is an external conjugacy between the parabolic-like maps.

Proof. Let $(X_{ni}, \rho_{(n-1)i}, \pi_{(n-1)i}, f_{ni})_{n\geq 1, i=1,2}$ be as in 2.3.2. Let us set $\phi_0 = \overline{\phi}$ and define recursively $\phi_n = \rho^{-1}_{(n-1)2} \circ \phi_{n-1} \circ \rho_{(n-1)1} : X_{n1} \to X_{n2}$.

$$\begin{array}{ccc} X_{n1} & \xrightarrow{\phi_n} & X_{n2} \\ \Big\downarrow \rho_{(n-1)1} & & \Big\downarrow \rho_{(n-1)2} \\ X_{(n-1)1} & \xrightarrow{\phi_{n-1}} & X_{(n-1)2} \end{array} \tag{2.6}$$

Then every $\phi_n : X_{n1} \to X_{n2}$ thus defined is an isomorphism and a conjugacy between f_{n1} and f_{n2}.

$$\begin{array}{ccc} A'_{n1} \subset X_{n1} & \xrightarrow{f_{n1}} & B_{n1} \subset X_{n1} \\ \Big\downarrow \phi_n & & \Big\downarrow \phi_n \\ A'_{n2} \subset X_{n2} & \xrightarrow{f_{n2}} & B_{n2} \subset X_{n2} \end{array} \tag{2.7}$$

Thus the family of isomorphisms ϕ_n induces an isomorphism $\Phi : T_1 \cup T'_1 \to T_2 \cup T'_2$ compatible with dynamics, and thus the external maps h_1 and h_2 are real-analytically conjugated. $\square$

Proposition 2.4.4. *Let $f : U' \to U$ and $g : V' \longrightarrow V$ be two parabolic-like mappings of degree d with connected Julia sets. If they are hybrid and externally equivalent, then they are holomorphically equivalent.*

Proof. Let $\varphi : A \to B$ be a hybrid equivalence between f and g, and $\psi : (A_1 \cup A'_1) \setminus K_f \to (B_1 \cup B'_1) \setminus K_g$ an external equivalence between f and g. Let $h : W' \to W$ be an external map of f constructed from the Riemann map $\alpha : \mathbb{C} \setminus K_f \to \mathbb{C} \setminus \overline{\mathbb{D}}$. Let A_f be a topological disc compactly contained in $(A_1 \cup A_1') \cap A$ and such that $B_g = \phi(A_f)$ is compactly contained in $(B_1 \cup B_1')$. Call $B_f' = \psi^{-1}(A_f)$. The map $\beta = \alpha \circ \psi^{-1} : B_g \setminus K_g \to W \setminus \overline{\mathbb{D}}$ is an external equivalence between g and h.

Define the map $\Phi : A_f \to B_f'$ as:

$$\Phi(z) = \begin{cases} \varphi & \text{on } K_f \\ \psi & \text{on } A_f \setminus K_f \end{cases}$$

By construction the map $\Phi : A_f \to B_f'$ conjugates the maps f and g conformally on A_f and quasiconformally with $\overline{\partial}\Phi = 0$ on K_f. We want to prove that the map Φ is holomorphic. By Rickmann Lemma (see below) Φ is holomorphic if Φ is continuous. Thus we just need to prove that it is continuous.

Define $W_f = h(h^{-1}(\alpha(A_f \setminus K_f)) \cap \alpha(A_f \setminus K_f)) \subset \alpha(A_f \setminus K_f)$ and $W_f' = h^{-1}(W_f)$. The restriction $h : W_f' \to W_f$ is proper holomorphic and of degree d. The map $\chi := \beta \circ \varphi \circ \alpha^{-1} : W_f' \setminus \overline{\mathbb{D}} \to W \cup W' \setminus \overline{\mathbb{D}}$ is a quasi-conformal homeomorphism (into its image) which autoconjugates h on $\Omega_h' \cup \gamma_{\pm h} \setminus \gamma_{\pm h}(0)$.

Applying the strong reflection principle with respect to the unit circle, we obtain a quasiconformal homeomorphism (into its image) $\widetilde{\chi} : \widetilde{W_f'} \to W \cup W'$, which autoconjugates h on $\Omega_h' \cup \gamma_h(0)$ (where $\widetilde{W_f'}$ is the set given by W_f', union its reflection with respect to the unit disc, union $\mathbb{S}^1$). Thus the restriction $\widetilde{\chi} : \mathbb{S}^1 \to \mathbb{S}^1$ is a quasisymmetric autoconjugacy of h on the unit circle. The preimages of the parabolic fixed point $z = 1$ are dense in $\mathbb{S}^1$. Thus an autoconjugacy of h on the unit circle is the identity. Therefore $\widetilde{\chi}\,|_{\mathbb{S}^1} = Id$.

Since the map $\widetilde{\chi} : W_f' \setminus \mathbb{D} \to W \setminus \mathbb{D}$ is a quasiconformal homeomorphism which coincides with the identity on $\mathbb{S}^1$, the hyperbolic distance between a point near $\mathbb{S}^1$ and its image is uniformly bounded, i.e. $\exists M > 0$ and $r > 1$ such that:

$$\forall z\,,\ 1/r < |z| < r,\ \ d_{(W \cup W') \setminus \mathbb{D}}(z, \beta \circ \varphi \circ \alpha^{-1}(z)) \leq M.$$

Since α and β are isometries, we obtain

$$d_{A_f \setminus K_f}(\beta^{-1} \circ \alpha(z), \varphi(z)) \leq M \text{ for } z \notin K_f,\ z \text{ in a neighborhood of } K_f.$$

Then $\beta^{-1} \circ \alpha(z)$ and $\varphi(z)$ converge to the same value as z converges to J_f, i.e. $\beta^{-1} \circ \alpha$ extends continuously to J_f by $\beta^{-1} \circ \alpha(z) = \phi(z),\ z \in J_f$. Thus Φ is continuous. The results follows by Rickmann Lemma (for a proof of Rickmann Lemma we refer to [DH], Lemma 2 pg. 303):

Lemma 2.4.2. ***Rickmann*** *Let $U \subset \mathbb{C}$ be open, $K \subset U$ be compact, $\phi : U \to \mathbb{C}$ and $\Phi : U \to \mathbb{C}$ be two maps which are homeomorphisms onto their images. Suppose that ϕ is quasi-conformal, that Φ is quasi-conformal on $U \setminus K$ and that $\Phi = \phi$ on K. Then Φ is quasiconformal and $D\Phi = D\phi$ almost everywhere on K.*

□

We can now prove the main statement of this section:

Theorem 2.4.5. *Let (f, U, U', γ_f) be a parabolic-like mapping of some degree $d > 1$, and $h : \mathbb{S}^1 \to \mathbb{S}^1$ be a parabolic external map of the same degree d. Then there exists a parabolic-like mapping (g, V, V', γ_g) which is hybrid equivalent to f and whose external class is $[h]$.*

Throughout this proof we assume, in order to simplify the notation, U and U' with C^1 boundaries (if U and U' do not have C^1 boundaries we consider a parabolic-like restriction of (f, U, U', γ_f) with C^1 boundaries).

Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be a parabolic external map of degree $d > 1$, z_* be its parabolic fixed point and $h : W' \longrightarrow W$ an extension degree d covering (see Definition 2.3.10). Define $B = W \cup \mathbb{D}$ and $B' = W' \cup \mathbb{D}$.

We are going to construct now dividing arcs $\widetilde{\gamma} : [-1, 1] \to B \setminus \mathbb{D}$ for h, such that on $\widetilde{\gamma}$ the dynamics of h is conjugate to the dynamics of f.

Let h_f be an external map of f, z_1 its parabolic fixed point, $h_f : W_f' \to W_f$ an extension degree d covering (where W_f, W_f' are neighborhoods of $\mathbb{S}^1$ in $\mathbb{C}$) and α an external equivalence between f and h_f. The dividing arcs $\gamma_{h_f\pm}$ are tangent to $\mathbb{S}^1$ at the parabolic fixed point z_1, and they divide W_f, W_f' in Δ_W, Ω_W and Δ_W', Ω_W' respectively (see 2.3.3).

Let $\Xi_{h_f\pm}$ be repelling petals for the parabolic fixed point z_1 which intersect the unit circle and $\phi_\pm : \Xi_{h_f\pm} \to \mathbb{H}_-$ be Fatou coordinates with axis tangent to the unit circle at the parabolic fixed point z_1. On the other hand, let $\Xi_{h\pm}$ be repelling petals for the parabolic fixed point z_* of h which intersect the unit circle and $\widetilde{\phi}_\pm : \Xi_{h\pm} \to \mathbb{H}_-$ be Fatou coordinates with axis tangent to the unit circle at the parabolic fixed point z_*.

Define

$$\widetilde{\gamma}_+ = \widetilde{\phi}_+^{-1}(\phi_{h_f+}(\gamma_{h_f+}))$$

and

$$\widetilde{\gamma}_- = \widetilde{\phi}_-^{-1}(\phi_{h_f-}(\gamma_{h_f-})).$$

Since the arcs γ_{h_f+}, γ_{h_f-} are tangent to the unit circle at z_1 (see Prop. 2.3.1), the arcs $\widetilde{\gamma}_+$, $\widetilde{\gamma}_-$ are tangent to the unit circle at z_*. The arc $\widetilde{\gamma} = \widetilde{\gamma}_+ \cup \widetilde{\gamma}_-$

divides the set B into Ω_B, Δ_B (with $\mathbb{D} \in \Omega_B$) and the set B' into Ω'_B, Δ'_B (with $\mathbb{D} \in \Omega'_B$).

Define the map $\widetilde{\phi}^{-1} \circ \phi_{h_f} : \gamma_{h_f} \to \widetilde{\gamma}$ as follows:

$$\widetilde{\phi}^{-1} \circ \phi_{h_f}(z) = \begin{cases} \widetilde{\phi}_+^{-1} \circ \phi_{h_f+} & \text{on } \gamma_{h_f+} \\ \widetilde{\phi}_-^{-1} \circ \phi_{h_f-} & \text{on } \gamma_{h_f-} \end{cases}$$

Let z_0 be the parabolic fixed point of f, and define the map $\psi : \gamma_f \to \widetilde{\gamma}$ as follows:

$$\psi(z) = \begin{cases} \widetilde{\phi}_+^{-1} \circ \phi_{h_f+} \circ \alpha & \text{on } \gamma_{f+} \setminus \{z_0\} \\ \widetilde{\phi}_-^{-1} \circ \phi_{h_f-} \circ \alpha & \text{on } \gamma_{f-} \setminus \{z_0\} \\ z_* & \text{on } z_0 \end{cases}$$

The map $\psi : \gamma_f \to \widetilde{\gamma}$ is an orientation preserving homeomorphism, real-analytic on $\gamma_f \setminus \{z_0\}$, which conjugates the dynamics of f and h. Let $\psi_0 : \partial U \to \partial B$ be an orientation preserving C^1-diffeomorphism coinciding with ψ on $\gamma_f \cap \partial U$ (it exists because both U and B have smooth boundary).

Claim 2.4.1. *There exists a quasiconformal map $\Phi_\Delta : \Delta \to \Delta_B$ which extends to ψ on γ_f, and to ψ_0 on $\partial U \cap \partial\Delta$.*

Proof. It is sufficient to construct a quasiconformal map $\Phi_{\Delta_W} : \Delta_W \to \Delta_B$ which extends to $\widetilde{\phi}^{-1} \circ \phi_{h_f}$ on γ_{h_f} and to $\psi_0 \circ \alpha^{-1}$ on $\alpha(\partial U \cap \partial\Delta)$. Then we will set $\Phi_\Delta = \Phi_{\Delta_W} \circ \alpha$.

The set $\partial\Delta_W$ is a quasicircle, since it is a piecewise C^1 closed curves with non zero interior angles. Indeed, γ_{h_f+} and γ_{h_f-} are tangent to $\mathbb{S}^1$ at the parabolic fixed point z_1 (see Prop. 2.3.1), and we can assume the angles between $\gamma_{h_f\pm}$ and ∂W_f, $\partial W'_f$ positive (we may take parabolic-like restrictions). The same argument shows that $\partial\Delta_B$ is a quasicircle.

Let $\Phi_f : \Delta_W \to \mathbb{D}$, $\Phi_h : \Delta_B \to \mathbb{D}$ be Riemann maps, and let $\Psi_f : \mathbb{D} \to \Delta_W$, $\Psi_h : \mathbb{D} \to \Delta_B$ be their inverse. By the Carathodory theorem the maps Ψ_f, Ψ_h extend continuously to the boundaries, and since $\partial\Delta_W$, $\partial\Delta_B$ are quasicircles, the extensions $\Psi_f : \mathbb{S}^1 \to \partial\Delta_W$, $\Psi_h : \mathbb{S}^1 \to \partial\Delta_B$ are quasisymmetric. Define the map $\widetilde{\Phi}_0 : \mathbb{S}^1 \to \mathbb{S}^1$ as follows:

$$\widetilde{\Phi_0}(z) = \begin{cases} \Psi_h^{-1} \circ \widetilde{\phi}^{-1} \circ \phi_{h_f} \circ \Psi_f & \text{on } \Psi_f^{-1}(\gamma_{h_f}) \\ \Psi_h^{-1} \circ \psi_0 \circ \alpha^{-1} \circ \Psi_f & \text{on } \Psi_f^{-1}(\partial\Delta_W \cup \partial W_f) \end{cases}$$

The map $\widetilde{\Phi}_0 : \mathbb{S}^1 \to \mathbb{S}^1$ is quasisymmetric, because the extensions of $\Psi_{h|}$ and Ψ_f to the unit circle are quasisymmetric, α is conformal, the map ψ_0 is a C^1-diffeomorphism and the proof of Prop. 2.3.11(1) shows that the map $\widetilde{\phi}^{-1} \circ \phi_{h_f} : \gamma_{h_f} \to \widetilde{\gamma}$ is quasisymmetric. Hence it extends by the Douady-Earle extension (see [DE]) to a quasiconformal map $\widetilde{\phi}_0 : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ which is a real-analytic diffeomorphism on $\mathbb{D}$. Thus $\Phi_{\Delta_W} := \Psi_h \circ \widetilde{\phi}_0 \circ \Phi_f$ is a quasiconformal map between $\overline{\Delta_W}$ and $\overline{\Delta_B}$, which is a real-analytic diffeomorphism on Δ_W, and which coincides with $\widetilde{\phi}^{-1} \circ \phi_{h_f}$ on γ_{h_f} and to $\psi_0 \circ \alpha^{-1}$ on $\alpha(\partial U \cap \partial\Delta)$.

$\square$

Let us define $\widetilde{\Delta}_B = h(\Delta_B \cap \Delta'_B)$, $\widetilde{B} = \Omega_B \cup \widetilde{\gamma} \cup \widetilde{\Delta}_B$, $\widetilde{B}' = h^{-1}(\widetilde{B})$, $\widetilde{\Omega}'_B = \Omega'_B \cap \widetilde{B}'$, $\widetilde{\Delta}'_B = \Delta'_B \cap \widetilde{B}'$. On the other hand define $\widetilde{\Delta} = \Phi_\Delta^{-1}(\widetilde{\Delta}_B)$, $\widetilde{\Delta'} = \Phi_\Delta^{-1}(\widetilde{\Delta}'_B)$, $\widetilde{U} = (\Omega \cup \gamma_f \cup \widetilde{\Delta}) \subset U$.

Consider

$$\widetilde{f}(z) = \begin{cases} \Phi_\Delta^{-1} \circ h \circ \Phi_\Delta & \text{on } \widetilde{\Delta'} \\ f & \text{on } \Omega' \cup \gamma_f \end{cases}$$

Define $\widetilde{U'} = \widetilde{f}^{-1}(\widetilde{U})$, and $\widetilde{\Omega}' = \widetilde{U'} \cap \Omega'$. The map $\widetilde{f} : \widetilde{U'} \to \widetilde{U}$ is a degree d proper and quasiregular map which coincides with f on $(\widetilde{\Omega}' \cup \gamma_f) \subset (\Omega' \cup \gamma_f)$. Define $\widehat{U'} = f^{-1}(\widetilde{U})$, $\widehat{\Delta'} = \Delta' \cap \widehat{U'}$ and $\widehat{\Omega}' = \Omega' \cap \widehat{U'}$. Then $(f, \widehat{U'}, \widetilde{U}, \gamma_f)$ is a parabolic-like restriction of (f, U', U', γ_f), and $\widehat{\Omega}' = \widetilde{\Omega}'$.

Set $Q_f = \Omega \setminus \widetilde{\Omega}'$, $Q_h = \Omega_B \setminus \widetilde{\Omega}'_B$. Let $\bar{\psi}_0 : \partial\widetilde{U} \to \partial\widetilde{B}$ be an orientation preserving C^1-diffeomorphism coinciding with ψ_0 on $\partial\Omega$, and let $\psi_1 : \partial\widetilde{U'} \to \partial\widetilde{B}'$ a lift of $\bar{\psi}_0 \circ \widetilde{f}$ to h.

Claim 2.4.2. *There exists a quasiconformal map $\widetilde{\psi} : U \setminus \widetilde{\Omega} \to B \setminus \widetilde{\Omega}_B$ such that the almost complex structure σ defined as:*

$$\sigma(z) = \begin{cases} \sigma_0 & \textit{on } K_f \\ \sigma_1 = \widetilde{\psi}^*(\sigma_0) & \textit{on } U \setminus \widetilde{\Omega} \\ (\widetilde{f}^n)^* \sigma_1 & \textit{on } \widetilde{f}^{-n}(Q_f \cup \widetilde{\Delta}) \end{cases}$$

is bounded and $\widetilde{f}$-invariant.

Proof. Let us start by constructing a quasiconformal map Ψ_R between the topological rectangles $\overline{Q_f}$ and $\overline{Q_h}$ which agrees with ψ on γ_f, with ψ_0 on ∂U and with ψ_1 on $\partial\widetilde{U'}$.

Let us call $a = \overline{Q_f} \cap \gamma_+$, $b = \overline{Q_f} \cap \partial U$, $c = \overline{Q_f} \cap \gamma_-$ and $d = \overline{Q_f} \cap \partial U'$ (see Fig. 2.11). Let Ψ_f, Ψ_h be the unique conformal maps sending $\overline{Q_f}$ and $\overline{Q_h}$ respectively onto straight rectangles. Define the orientation preserving piecewise C^1-diffeomorphism $\widetilde{\Psi}_0 : \Psi_f(\partial Q_f) \to \Psi_h(\partial Q_h)$ as follows:

$$\widetilde{\Psi_0}(z) = \begin{cases} \Psi_h \circ \psi_+ \circ \Psi_f^{-1} & \text{on } \Psi_f(a) \\ \Psi_h \circ \psi_0 \circ \Psi_f^{-1} & \text{on } \Psi_f(b) \\ \Psi_h \circ \psi_- \circ \Psi_f^{-1} & \text{on } \Psi_f(c) \\ \Psi_h \circ \psi_1 \circ \Psi_f^{-1} & \text{on } \Psi_f(d) \end{cases}$$

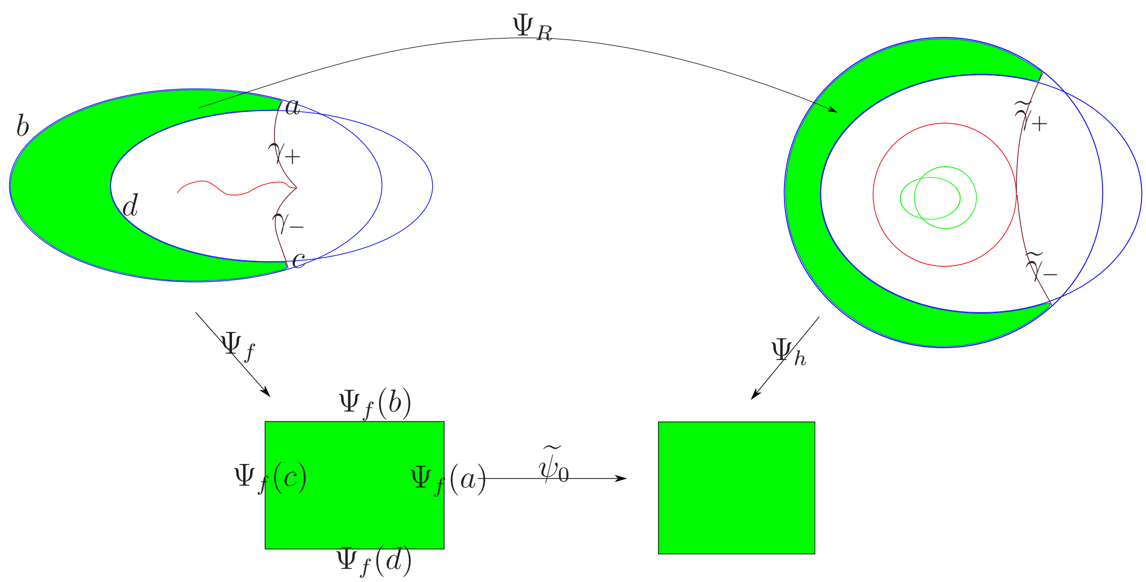


Figure 2.11: Construction of the quasiconformal map Ψ_R between the topological rectangles Q_f and Q_h.

Let $\widetilde{\psi}_0 : \Psi_f(\overline{Q_f}) \to \Psi_h(\overline{Q_h})$ be a quasiconformal extension (see [BF] pg.48). Then the map $\widetilde{\psi}_0 : \Psi_f(\overline{Q_f}) \to \Psi_h(\overline{Q_h})$ is a C^1-diffeomorphism and therefore the map $\Psi_R := \Psi_h^{-1} \circ \widetilde{\psi}_0 \circ \Psi_f : \overline{Q_f} \to \overline{Q_h}$ is a C^1-diffeomorphism. In particular it is a quasiconformal map.

Let $\widetilde{\psi} : U \setminus \widetilde{\Omega} \to B \setminus \widetilde{\Omega}_B$ be the quasiconformal homeomorphism defined as follows:

$$\widetilde{\psi}(z) = \begin{cases} \psi & \text{on } \gamma_f \\ \Psi_R & \text{on } Q_f \\ \Phi_\Delta & \text{on } \Delta \end{cases}$$

Define on U a new almost complex structure σ defined as follows:

$$\sigma(z) = \begin{cases} \sigma_0 & \text{on } K_f \\ \sigma_1 = \widetilde{\psi}^*(\sigma_0) & \text{on } U \setminus \widetilde{\Omega}' \\ (\widetilde{f}^n)^*\sigma_1 & \text{on } \widetilde{f}^{-n}(Q_f \cup \widetilde{\Delta}) \end{cases}$$

The almost complex structure σ is bounded and $\widetilde{f}$-invariant by construction.

$\square$

By the Measurable mapping theorem, there exists a quasiconformal map $\varphi : U \to \mathbb{D}$ such that $\varphi^*\sigma_0 = \sigma$.

Let

$$g := \varphi \circ \widetilde{f} \circ \varphi^{-1} : \varphi(\widetilde{U'}) \to \varphi(\widetilde{U}) \subset \mathbb{D}.$$

Let us call $V' = \varphi(\widetilde{U'})$, $V = \varphi(\widetilde{U})$, $\gamma_{g+} = \varphi(\gamma_{f+})$ and $\gamma_{g-} = \varphi(\gamma_{f-})$. Then $(g, V, V', \gamma_{g+}, \gamma_{g-})$ is a parabolic-like map of the same degree as f, and $\varphi : \widetilde{U} \to V$ is a hybrid conjugacy between f and g. Indeed, since $\widetilde{f}$ coincides with f on $\widetilde{\Omega}' \cup \gamma_f$, the map φ is a quasiconformal conjugacy between f and g, and, by construction, $\varphi^*\sigma_0 = \sigma_0$ on K_f.

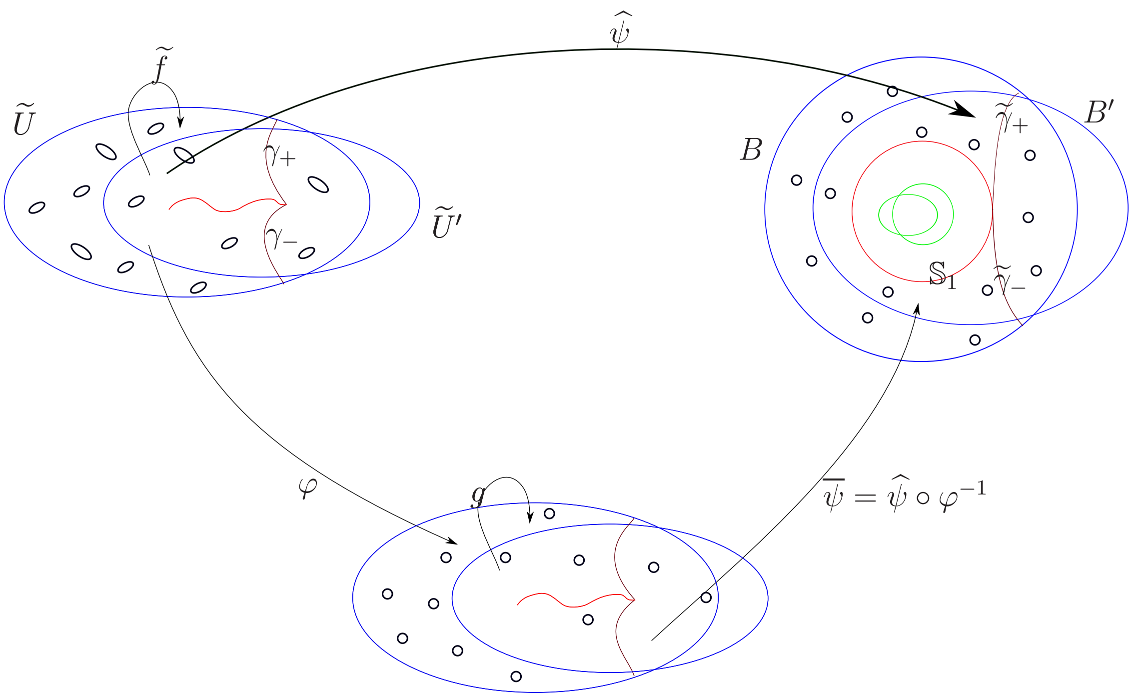


Figure 2.12: The map $\overline{\psi} = \widehat{\psi} \circ \varphi^{-1}$ is an external conjugacy between g and h.

If K_f is connected, define the map $\widehat{\psi} : \widetilde{U} \setminus K_f \to B \setminus \overline{\mathbb{D}}$ as follows:

$$\widehat{\psi}(z) = \begin{cases} \widetilde{\psi} & \text{on } U \setminus \widetilde{\Omega}' \\ h^{-n} \circ \widetilde{\psi} \circ \widetilde{f}^n & \text{on } \widetilde{f}^{-n}(Q_f \cup \widetilde{\Delta}) \end{cases}$$

Then $\widehat{\psi}$ is a quasiconformal map. Then the quasiconformal map $\overline{\psi} = \widehat{\psi} \circ \varphi^{-1} : (V \cup V') \setminus K_g \to B \setminus \overline{\mathbb{D}}$ is an external conjugacy between g and h, since it is holomorphic (indeed $(\widehat{\psi} \circ \varphi^{-1})^* \sigma_0 = \sigma_0$) and $\overline{\psi} \circ g = h \circ \overline{\psi}$ on $(V \cup V') \setminus K_g$ by construction (see Fig. 2.12).

If K_f is not connected, let $V_f \approx \mathbb{D}$ be a full relatively compact connected subset of $\widetilde{U}$, containing $\overline{\widetilde{\Omega}'}$, the critical values of $\widetilde{f}$ and such that $\widetilde{f} : \widetilde{f}^{-1}(V_f) \to V_f$ is a parabolic-like restriction of $(\widetilde{f}, \widetilde{U}, \widetilde{U'}\, \gamma_f)$. Call $L = \widetilde{f}^{-1}(\overline{V}_f) \cap \overline{\widetilde{\Omega}'}$.

Define the map $\widehat{\psi} : (\widetilde{U} \cup \widetilde{U'}) \setminus L \to B \setminus \overline{\mathbb{D}}$ as follows:

$$\widehat{\psi}(z) = \begin{cases} \widetilde{\psi} & \text{on } U \setminus \widetilde{\Omega}' \\ h^{-n} \circ \widetilde{\psi} \circ \widetilde{f}^n & \text{on } (\widetilde{U} \cup \widetilde{U'}) \setminus L \end{cases}$$

Let $V_g \approx \mathbb{D}$ be a full relatively compact connected subset of V containing $\overline{\Omega}'_g$, the critical values of g and such that $g : g^{-1}(V_g) \to V_g$ is a parabolic-like restriction of (g, V, V', γ_g). Call $M = g^{-1}(\overline{V}_g) \cap \overline{\Omega}'_g$, and consider the restriction $\varphi : (\widetilde{U} \cup \widetilde{U'}) \setminus L \to (V \cup V') \setminus M$.

Then the map $\overline{\psi} = \widehat{\psi} \circ \varphi^{-1} : (V \cup V') \setminus M \to B \setminus \overline{\mathbb{D}}$ is an external conjugacy between g and h (see Lemma 2.4.1).

2.5 The Straightening Theorem

Polynomial-like maps can be straightened to polynomials, while the aim of this chapter is to prove that parabolic-like maps can be straightened to rational maps with a parabolic fixed point of multiplier 1. The filled Julia set K_P of a polynomial $P : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is defined as the complement of the basin of attraction of infinity, which is a completely invariant Fatou component. On the other hand, the filled Julia set is not defined in the literature for general rational maps. However, for $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of degree d with a completely invariant Fatou component Λ we may define the filled Julia set as

$$K_R = \overline{\mathbb{C}} \setminus \Lambda.$$

In this case $R : \Lambda \to \Lambda$ is a proper holomorphic degree d map. Note that a degree d map can have up to 2 completely invariant Fatou components Λ_1, Λ_2 (since a degree d map defined on the Riemann sphere has $2d-2$ critical points, and a completely inveriant Fatou component has at least $d-1$ critical points). In the case R has precisely 1 completely invariant component

Λ, the filled Julia set $K_R = \overline{\mathbb{C}} \setminus \Lambda$ is well defined. In the case R has 2 such Fatou components Λ_1, Λ_2, there are 2 possibilities for the filled Julia set, hence we need to make a choice. After choosing a completely invariant component Λ_*, the filled Julia set $K_R = \overline{\mathbb{C}} \setminus \Lambda_*$ is well defined. Note that in this case both Λ_1, Λ_2 are isomorphic to a disc, and the Julia set $J_R = \overline{\mathbb{C}} \setminus (\Lambda_1 \cup \Lambda_2)$ is a Jordan curve.

Equivalently, let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree d. The map f has a *parabolic-like restriction* if there exist open connected sets $U,\ U' \approx \mathbb{D}$ and dividing arcs $\gamma_\pm$ such that $(f, U', U, \gamma_+, \gamma_-)$ is a parabolic-like map of some degree $d' \leq d$. If $d' = d$, i.e. if U contains $d-1$ critical points of f, the parabolic-like restriction is *maximal*. Then we consider as filled Julia set of $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ the filled Julia set of its maximal parabolic-like restriction. In the remainder of the thesis we are considering maximal parabolic-like restrictions without further reference.

For example, let us consider the map $h_2 = \frac{z^2+\frac{1}{3}}{1+\frac{z^2}{3}}$. This map has a parabolic fixed point at $z = 1$ with multiplier 1 and parabolic multiplicity 2, and simple critical points at $z = 0$ and at $z = \infty$. It has 2 completely invariant Fatou components, $\mathbb{D}$ and $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Since the map is symmetric with respect to the unit circle, which is an invariant set, the 2 possibilities for the filled Julia set $\overline{\mathbb{C}} \setminus \mathbb{D}$ and $\overline{\mathbb{D}}$ are equivalent. In the same way we can construct 2 equivalent maximal parabolic-like restrictions. In the examples (see 2.2.1) we chose the domain and range of the maximal parabolic-like restriction to be $U' = \{z : |z| < 1 + \epsilon\}$ (for some $\epsilon > 0$) and $U = h_2(U')$, and therefore $K_{h_2} = \overline{\mathbb{D}}$.

For a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with a maximal parabolic-like restriction, we consider as the external class of f the external class of its parabolic-like restriction, and we say that two holomorphic maps f, $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ are externally conjugate if their parabolic-like restrictions are externally conjugate.

Remarks 2.5.1. *We can take parabolic-like restrictions of parabolic-like maps without changing the filled Julia set (see 2.4), and thus there exist many different equivalent parabolic-like restrictions of a map. This will be really useful in the proof of Prop. 2.5.1.*

2.5.1 The family $Per_1(1)$

Let Rat_2 be the space of all rational maps $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of degree 2. The quotient of Rat_2 modulo möbius conjugacy is the moduli space M_2. Let

$Per_1(1) \subset M_2$ be the set of möbius conjugacy classes of quadratic rational maps which have a fixed point with multiplier 1, i.e.

$$Per_1(1) = \{[f] \in M_2 \,|\, f \text{ has a parabolic fixed point with multiplier } 1\}.$$

If we fix the parabolic fixed point to be infinity and the critical points to be ± 1, then we obtain

$$Per_1(1) = \{[P_A] \,|\, P_A(z) = z + 1/z + A,\ A \in \mathbb{C}\}.$$

For $A \in \mathbb{C}$ the map $P_A = z + 1/z + A$ has a parabolic fixed point at ∞ with multiplier 1, a fixed point at $1/A$ with multiplier $B = 1 - A^2$ and two critical points at ± 1. Note that if P_{A_1} and P_{A_2} are holomorphically conjugate, then $(A_1)^2 = (A_2)^2$. Indeed, a Möbius transformation which conjugates P_{A_1} and P_{A_2} fixes the parabolic fixed point $z = \infty$ and its preimage $z = 0$, and it can fix or interchange the critical points $z = 1$ and $z = -1$. Hence there exist just two possible conformal conjugacies between P_{A_1} and P_{A_2}, which are the Identity and the map $z \to -z$. Therefore a class $[P_A]$ consists of two maps, i.e.

$$[P_A] = \{P_A,\ P_{-A}\}.$$

Proposition 2.5.1. *For every $A \in \mathbb{C}$ the external class of P_A is given by the class of $h_2 = \frac{z^2+\frac{1}{3}}{1+\frac{z^2}{3}}$.*

Proof. The map $\phi(z) = \frac{z+1}{z-1}$ is a conformal conjugacy between the maps $P_0(z) = z + 1/z$ and $h_2 = \frac{3z^2+1}{3+z^2}$. Therefore, in order to prove that h_2 is an external map of P_A, it is sufficient to prove that P_0 with filled Julia set $\overline{\mathbb{H}}_- = \phi(\overline{\mathbb{D}})$ is externally equivalent to P_A, for $A \in \mathbb{C}$.
Replacing A by $-A$ if necessary, we can assume that $z = 1$ is the first critical point attracted by ∞, which basin Λ defines the filled Julia set $K_{P_A} = \overline{\mathbb{C}} \setminus \Lambda$. Let Ξ^0 be an attracting petal of P_0 containing the critical value $z = 2$, and let $\Phi_0 : \Xi^0 \to \mathbb{H}_+$ be the incoming Fatou coordinates of P_0 normalized by $\Phi_0(2) = 1$. Let Ξ^A be the attracting petal of P_A and let $\Phi_A : \Xi^A \to \mathbb{H}_+$ be the incoming Fatou coordinate of P_A with $\Phi_A(2 + A) = 1$.

Let us contruct an external equivalence first in the case K_{P_A} is connected. Define $\eta = \Phi_A^{-1} \circ \Phi_0 : \Xi^0 \to \Xi^A$. The map $\eta(z)$ is a conformal conjugacy between P_0 and P_A on Ξ^0. Defining Ξ^0_{-n}, $n > 0$ as the connected component of $P_0^{-n}(\Xi^0)$ containing Ξ^0, and Ξ^A_{-n}, $n > 0$ as the connected component of $P_A^{-n}(\Xi^A)$ containing Ξ^A, we can lift the map η to $\eta_n : \Xi^0_{-n} \to \Xi^A_{-n}$. Since K_A is connected by interated lifting of η we obtain an external conjugacy $\overline{\eta} : \widehat{\mathbb{C}} \setminus K_{P_0} \to \widehat{\mathbb{C}} \setminus K_{P_A}$ between P_0 and P_A. Thus $h_2 = \frac{3z^2+1}{3+z^2}$ is an external map for P_A.

In the case K_{P_A} is not connected the map $\eta(z)$ is a conformal conjugacy between P_0 and P_A on the region delimited by the Fatou equipotential passing through $z = 1$.

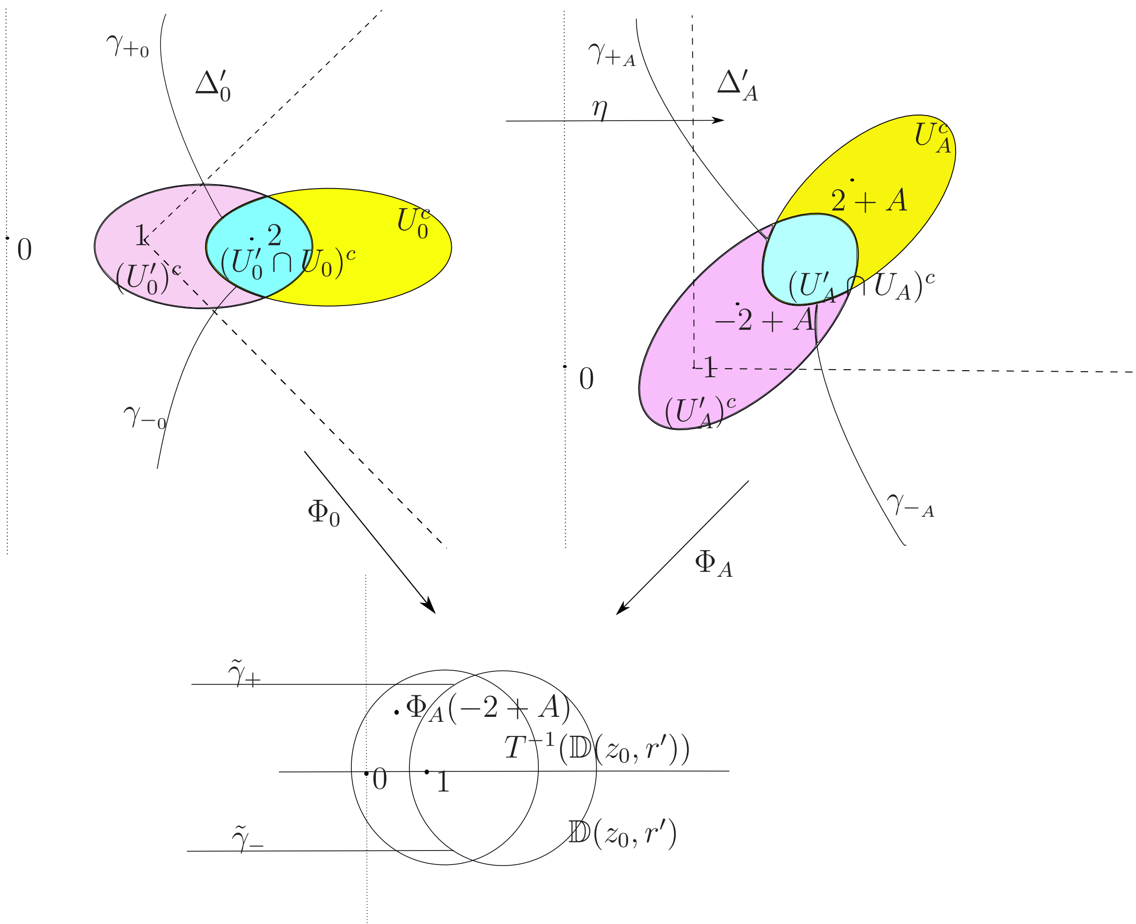


Figure 2.13: The construction of the parabolic-like restrictions of P_0 and P_A which allow us to extend the map $\eta(z)$ to an external conjugacy between them. In the picture we are assuming the critical value $z = -2 + A$ in $\Omega_A \setminus \Omega'_A$. In this case the critical value $z = -2 + A$ belongs to the attracting petal Ξ_A.

We are going to construct parabolic-like restrictions $(P_0, U_0, U'_0, \gamma_{+_0}, \gamma_{-_0})$ and $(P_A, U'_A, U_A, \gamma_{+_A}, \gamma_{-_A})$ of the maps P_0 and P_A respectively and extend the map $\eta(z)$ to an external conjugacy between them. Since the critical point $z = 1$ is the first attracted by infinity for both the maps P_0 and P_A, it cannot belong to the domains U'_0, U'_A of their parabolic-like restrictions, but it may belong to the codomains U_0, U_A. On the other hand the critical point $z = -1$

belongs to both the domains of the parabolic-like restrictions of P_0 and P_A, and in particular it must belong to Ω'_0 and Ω'_A.

Let us denote by $\widehat{\Phi_A}$, $\widehat{\Phi_0}$ the Fatou coordinate of P_A, P_0 respectively extended to the whole basin of attraction of ∞ by iterated lifting. Note that $\widehat{\Phi_A}$, $\widehat{\Phi_0}$ have univalent inverse branches $\psi_A : \mathbb{C} \setminus \{z = x+iy | x < 0 \wedge y \in [0, Im\widehat{\Phi_A}(-2+A)]\} \to \widehat{\Xi}_A$ and $\psi_0 : \mathbb{C} \setminus \mathbb{R}_- \to \widehat{\Xi}_0$, and the map $\eta = \psi_A \circ \widehat{\Phi}_0 : \psi_0^{-1}(\mathbb{C} \setminus \{z = x+iy | x < 0 \wedge y \in [0, Im\widehat{\Phi_A}(-2+A)]\}) \to \widehat{\Xi}_A$ is a biholomorphic extension of η conjugating dynamics.

Choose $r > max\{1 + Im(\widehat{\Phi_A}(A-2)),\ 2\}$ and $z_0,\ r < z_0 < r+1$ such that $A - 2 \notin \Phi_A^{-1}(\overline{\mathbb{D}(z_0, r)})$. Then for $r < r' < z_0$ with r' sufficiently close to r we have $A - 2 \notin \Phi_A^{-1}(\mathbb{D}(z_0, r'))$.

Let $\widetilde{\gamma}_+$, $\widetilde{\gamma}_-$ be horizontal lines, symmetric with respect to the real axis, starting at $-\infty$ and landing at $\partial\mathbb{D}(z_0, r)$, such that the point $\widehat{\Phi_A}(A-2)$ is contained in the strip between them (see Fig. 2.13) and they do not leave the disk $T^{-1}(\mathbb{D}(z_0, r))$ (i.e. the disk of radius r and center $z_1 = z_0 - 1$) after having entered to it.

Define $U_0 = (\Phi_0^{-1}(\mathbb{D}(z_0, r))^c$, $U'_0 = P_0^{-1}(U_0)$, $\gamma_{+_0} = \psi_0(\widetilde{\gamma}_+)$, and $\gamma_{-_0} = \psi_0(\widetilde{\gamma}_-)$. In the same way define $U_A = (\Phi_A^{-1}(\mathbb{D}(z_0, r))^c$, $U'_A = P_A^{-1}(U_A)$, $\gamma_{+_A} = \psi_A(\widetilde{\gamma}_+)$, and $\gamma_{-_A} = \psi_A(\widetilde{\gamma}_-)$.

Then the parabolic-like restriction of P_0 we consider is $(P_0, U_0, U'_0, \gamma_{+_0}, \gamma_{-_0})$, and the parabolic-like restriction of P_A we consider is $(P_A, U_A, U'_A, \gamma_{+_A}, \gamma_{-_A})$.

The arc $\gamma_{-_A} \cup \gamma_{+_A}$ divides U'_A, U_A into Ω'_A, Δ'_A and Ω_A, Δ_A respectively (with $\Omega'_A \subset\subset U_A$, $\Omega'_A \subset \Omega_A$ and $\Delta'_A \cap \Delta_A \neq \emptyset$). Note that, by construction, the map η is a conformal conjugacy between P_0 and P_A on Δ'_0.

In order to obtain an external conjugacy we need η to be defined on an annulus. Thus we need to extend η to some annulus containing the boundary of U'_0.

Define $D_0 = \Phi_0^{-1}(\mathbb{D}(z_0,\ r')) \subset \Xi_0$, $D'_0 = P_0^{-1}(D_0)$, $D_A = \Phi_A^{-1}(\mathbb{D}(z_0,\ r')) \subset \Xi_A$, and $D'_A = P_A^{-1}(D_A)$ (see Fig. 2.14). Then the restriction $\eta : D_0 \setminus (U_0)^c \to D_A \setminus (U_A)^c$ is a holomorphic conjugacy between P_0 and P_A. Since $-2 \notin D_0$, $-2 + A \notin D_A$, the restrictions $P_0 : D'_0 \setminus (U'_0)^c \to D_0 \setminus (U_0)^c$ and $P_A : D'_A \setminus (U'_A)^c \to D_A \setminus (U_A)^c$ are degree 2 covering. Then we can lift the map η to a biholomorphic map $\eta : (D'_0 \setminus (U'_0)^c) \cup \Delta'_0 \to (D'_A \setminus (U'_A)^c) \cup \Delta'_A$ wich conjugates dynamics.

Let us define the sets $V_0 = P_0(\overline{(D'_0)^c})$ and $V_A = P_A(\overline{(D'_A)^c})$, hence $L = \overline{\Omega}'_0 \setminus D'_0$ and $M = \overline{\Omega}'_A \setminus D'_A$. Then L and M are compact subsets of U'_0, U'_A respectively, containing the critical point $z = -1$ for P_0, P_A respectively and such that $P_0 : (D'_0)^c \to P_0((D'_0)^c)$, $P_A : (D'_A)^c \to P_A((D'_A)^c)$ are parabolic-like restrictions of $(P_0, U_0, U'_0, \gamma_{+_0}, \gamma_{-_0})$ and $(P_A, U_A, U'_A, \gamma_{+_A}, \gamma_{-_A})$ respectively.

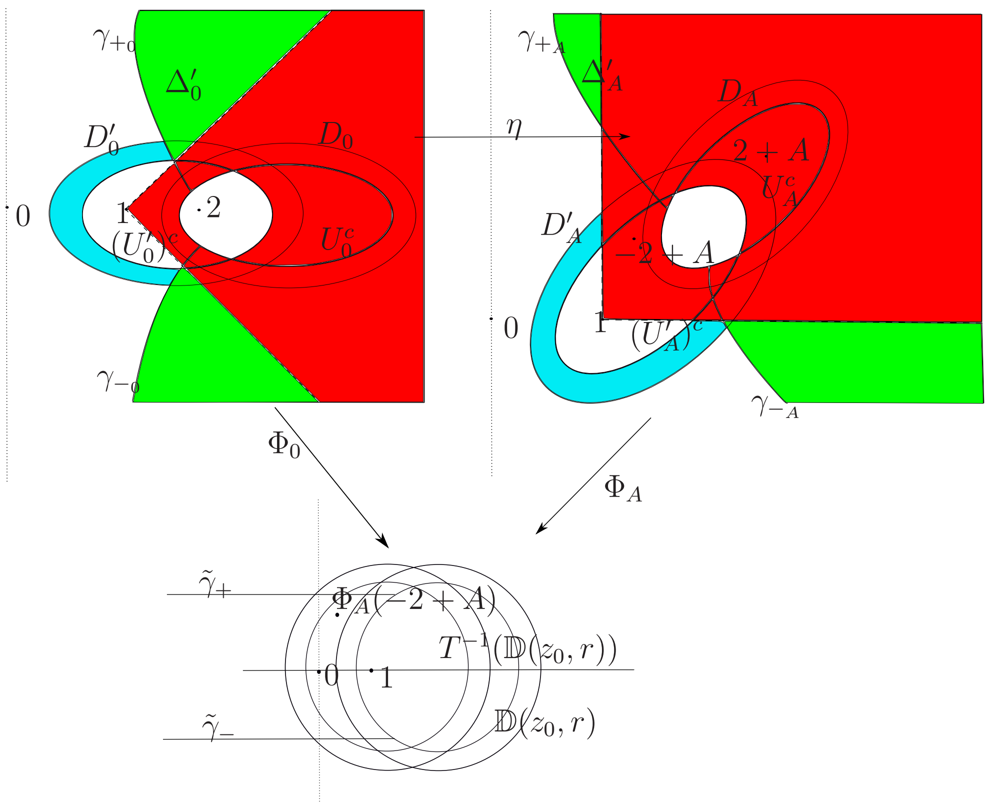


Figure 2.14: The construction of the external conjugacy η between the parabolic-like restriction of P_0 and the parabolic-like restriction of P_A. In the picture we are assuming the critical value $z = -2 + A$ in $\Omega_A \setminus \Omega'_A$.

Since the map $\eta : (U_0 \cup U'_0) \setminus L \to (U_A \cup U'_A) \setminus M$ is a biholomorphic extended conjugacy, the result follows applying the Lemma 2.4.1. $\square$

Proposition 2.5.2. *If $P_A = z + 1/z + A$ and $P_{A'} = z + 1/z + A'$ are hybrid conjugate and K_A is connected, then they are holomorphically conjugate, i.e. $A^2 = (A')^2$ and P_A and $P_{A'}$ are the two representatives of the same class in $Per_1(1)$.*

Proof. Since K_A and $K_{A'}$ are connected, the external conjugacies between P_A and $P_{A'}$ respectively and h_2 can be extended to the discs $\widehat{\mathbb{C}} \setminus K_A$ and $\widehat{\mathbb{C}} \setminus K_{A'}$ (see Prop. 2.5.1), i.e. there exist holomorphic conjugacies $\alpha : \widehat{\mathbb{C}} \setminus K_A \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and $\beta : \widehat{\mathbb{C}} \setminus K_{A'} \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ between P_A and $P_{A'}$ respectively and h_2. Therefore $\beta^{-1} \circ \alpha : \widehat{\mathbb{C}} \setminus K_{A'} \to \widehat{\mathbb{C}} \setminus K_{A'}$ is a holomorphic conjugacy between P_A and $P_{A'}$.

Let (P_A, U', U, γ) and $(P_{A'}, V', V, \gamma')$ be parabolic-like restrictions of P_A and $P_{A'}$ respectively, and let $\varphi : U \to V$ be a hybrid equivalence between them. Define the map $\Phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as follows:

$$\Phi(z) = \begin{cases} \varphi & \text{on } K_A \\ \beta^{-1} \circ \alpha & \text{on } \widehat{\mathbb{C}} \setminus K_A \end{cases}$$

The proof of Prop. 2.4.4 shows that the map $\Phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is holomorphic. Therefore Φ is a Möbius transformation. Hence $A^2 = (A')^2$ and P_A and $P_{A'}$ are the two representatives of the same class in $Per_1(1)$.

$\square$

2.5.2 The Straightening Theorem

Theorem 2.5.3. *Every parabolic-like mapping $f : U' \to U$ of degree 2 is hybrid equivalent to a member of the family $Per_1(1)$.*

Moreover, if K_f is connected, this member is unique.

Proof. Let $g : V' \to V$ be the map obtained from f and $h_2 = \frac{z^2 + \frac{1}{3}}{1 + \frac{z^2}{3}}$ by Prop. 2.4.5. Let $\overline{\psi}$ be an external conjugacy between the maps g and h_2. Let S be the Riemann surface obtained by gluing $V \cup V'$ and $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, by the equivalence relation identifying z to $\overline{\psi}(z)$, i.e.

$$S = (V \cup V') \coprod (\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}) / z \sim \overline{\psi}(z).$$

By the Uniformization theorem, S is isomorphic to the Riemann sphere. Consider the map

$$\widetilde{g}(z) = \begin{cases} g & \text{on } V' \\ h_2 & \text{on } \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \end{cases}$$

Since the map h_2 is the external map of g, the map $\tilde{g}$ is continuous and then holomorphic. Let $\widehat{\phi} : S \to \widehat{\mathbb{C}}$ be an isomorphisim that sends the parabolic fixed point of $\widetilde{g}$ to infinity, the critical point of $\widetilde{g}$ to $z = -1$, and the preimage of the parabolic fixed point of $\widetilde{g}$ to $z = 0$. Define $P_2 = \widehat{\phi} \circ \tilde{g} \circ \widehat{\phi}^{-1} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. The map P_2 is a holomorphic function hybrid conjugate to the map f. Let us show that P_2 is a member of a conjugacy class of

$$Per_1(1) = \{[P_A] \mid P_A(z) = z + 1/z + A, \; A \in \mathbb{C}\}$$

The map P_2 is holomorphic on the Riemann sphere and with degree 2, so it is a quadratic rational function. Moreover, by construction, it has a parabolic fixed point of multiplier 1 at $z = \infty$ with preimage $z = 0$, and it has a critical point at $z = -1$. Therefore $P_2 = P_A$ for some A.

The uniqueness of the class $[P_A]$ in the case K_f is connected follows from Prop. 2.5.2. Indeed, if $P_A = z + 1/z + A$ and $P_{A'} = z + 1/z + A'$ with $A \neq A'$ are hybrid conjugate to f and K_f is connected, then P_A and $P_{A'}$ are hybrid conjugate and K_A is connected. Hence by Prop. 2.5.2, P_A and $P_{A'}$ are the two representatives of the same class in $Per_1(1)$.

$\square$

Chapter 3

Analytic families of Parabolic-like maps

3.1 Introduction

By theorem 2.5.3 in chapter 2 if f is a parabolic-like map of degree $d = 2$, f is hybrid equivalent to a member of the family

$$Per_1(1) = \{[P_A] \,|\, P_A(z) = z + 1/z + A,\ A \in \mathbb{C}\},$$

and if K_f is connected this member is unique (up to holomorphic conjugacy).

Note that, if P_{A_1} and P_{A_2} are holomorphically conjugate, then $(A_1)^2 = (A_2)^2$. Indeed, a Möbius transformation which conjugates P_{A_1} and P_{A_2} fixes the parabolic fixed point $z = \infty$ and its preimage $z = 0$, and it can fix or interchange the critical points $z = 1$ and $z = -1$. Hence a class $[P_A]$ in $Per_1(1)$ contains two maps, i.e.

$$[P_A] = \{P_A,\ P_{-A}\}.$$

In the following we will refer to a quadratic rational map of the family $Per_1(1)$ as one of these representatives of its class.

The family $Per_1(1)$ is typically parametrized by $B = 1 - A^2$, which is the multiplier of the 'free' fixed point $z = -1/A$ of P_A. The connectedness locus of $Per_1(1)$ is called M_1.

Hence if $\mathbf{f} = (f_\lambda : U'_\lambda \to U_\lambda)$ is an analytic family of parabolic-like maps of degree 2, we can define a map

$$\chi : M_f \to M_1$$

$$\lambda \to B,$$

which associates to each λ the multiplier of the fixed point of the member $[P_A]$ hybrid equivalent to f_λ.

The aim of this chapter is indeed to prove that the map χ extends to a map defined on Λ, whose restriction to M_f, under suitable conditions (see Definition 3.5.7) is a ramified covering of $M_1 \setminus \{1\}$. The reason why the map χ extends to a ramified covering of $M_1 \setminus \{1\}$, instead of the whole of M_1, resides in the definition of analytic family of parabolic-like mappings (see 3.2.1), and it will be explained in section 3.2.1.

3.2 Definition

Definition 3.2.1. Let $\Lambda \subset \mathbb{C}$, $\Lambda \approx \mathbb{D}$ and let $\mathbf{f} = (f_\lambda : U'_\lambda \to U_\lambda)_{\lambda \in \Lambda}$ be a family of parabolic-like mappings. Set $\mathbf{U'} = \{(\lambda, z) |\ z \in U'_\lambda\}$, $\mathbf{U} = \{(\lambda, z) |\ z \in U_\lambda\}$, $\Omega'_f = \{(\lambda, z) |\ z \in \Omega'_\lambda\}$, $\Omega_f = \{(\lambda, z) |\ z \in \Omega_\lambda\}$ and $f(\lambda, z) = (\lambda,\ f_\lambda(z))$. Then $\mathbf{f}$ in an analytic family of parabolic-like maps if the following conditions are satisfied:

1. $\mathbf{U'}$,$\mathbf{U}$,Ω'_f and Ω_f are homeomorphic over Λ to $\Lambda \times \mathbb{D}$;

2. the projection from the closure of Ω'_f in $\mathbf{U}$ to Λ is proper;

3. the map $f : \mathbf{U'} \to \mathbf{U}$ is complex analytic and proper. In particular $f(\lambda, z)$ is continuous and holomorphic in (λ, z);

4. for each $\lambda \in \Lambda$ the map $f_\lambda : U'_\lambda \to U_\lambda$ is a parabolic-like map with *the same number of attracting petals in its filled Julia set*;

5. the dividing arcs move holomorphically, i.e. we have a holomorphic motion
$$\Phi : \Lambda \times \gamma_{\lambda_0} \to \mathbb{C};$$

6. the boundaries of the codomains move holomorphically and the motion defines a piecewise C^1-diffeomorphism with no cusps in z, i.e. we have a holomorphic motion
$$B : \Lambda \times \partial U_{\lambda_0} \to \mathbb{C}$$
which is a piecewise C^1-diffeomorphism with no cusps in z (for every fixed λ). Moreover, $B_\lambda(\gamma_{\lambda_0}(\pm 1)) = \gamma_\lambda(\pm 1)$.

Note that the fact that $\Phi : \Lambda \times \gamma_{\lambda_0} \to \gamma_\lambda$ is a holomorphic motion implies that the map Φ extends to a quasiconformal homeomorphism whose restriction $\Phi_\lambda : \gamma_{\lambda_0} \to \gamma_\lambda$ conjugates dynamics.

Notation. *As in the chapter 2, we will use through out this chapter both the notations*

- $\gamma_\lambda : [-1,1] \to \overline{U_\lambda}, \gamma_\lambda(0) = z_\lambda,$
- $\gamma_{\lambda+} : [0,1] \to \overline{U_\lambda},\ \ \gamma_{\lambda-} : [0,-1] \to \overline{U_\lambda},\ \ \gamma_{\lambda\pm}(0) = z_\lambda,\ \gamma_\lambda := \gamma_{\lambda+} \cup \gamma_{\lambda-}.$

Remarks about the definition

Note that we require all the maps in an analytic family of parabolic-like maps to have the same number of attracting petals in its filled Julia set (see 3.2.1 (4)). This condition is necessary to allow us to ask a *holomorphic motion of the dividing arcs* (see 3.2.1 (5)). Indeed, the dividing arcs for a parabolic-like map $f_{\hat{\lambda}}$ with no attracting petals in $K_{f_{\hat{\lambda}}}$ form a cusp at the parabolic fixed point. On the other hand, the dividing arcs for a parabolic-like map $f_{\tilde{\lambda}}$ with a positive number of petals in $K_{f_{\tilde{\lambda}}}$ form a positive angle on both the side of $K_{f_{\tilde{\lambda}}}$ and the side of $\Delta_{f_{\tilde{\lambda}}}$, and it is well known that there is no quasiconformal map mapping a cusp to a curve with positive angle.

Degree, Filled Julia set, Julia set and connectedness locus for analytic families of parabolic-like maps

The degree of the analytic family f_λ is independent of λ. Indeed, since the family f_λ depends holomorphically on λ, the degree depends continuously on the parameter, and since it is a natural number, it is constant, and therefore it is independent of λ. We call it the degree of $\mathbf{f}$.

For all $\lambda \in \Lambda$ let us call z_λ the parabolic-fixed point of f_λ, and let us set

- $K_\lambda = K_{f_\lambda},$
- $J_\lambda = J_{f_\lambda}$
- $\mathbf{K}_f = \{(\lambda, z) \mid z \in K_\lambda\}.$

The set $\mathbf{K}_f$ is closed in $\overline{\Omega'}_f$, and since the projection from the closure of Ω'_f in $\mathbf{U}$ to Λ is proper, the projection of $\mathbf{K}_f$ into Λ is proper.

Define

$$M_f = \{\lambda \mid K_\lambda \text{ is connected}\}.$$

3.2.1 Analytic families of parabolic-like maps of degree 2

The definition of analytic family of parabolic-like maps is generic, but in this chapter we are interested in proving that the map χ defined in the introduction is a ramified covering between M_f and $M_1 \setminus \{1\}$, hence in the remainder of this thesis we will consider analytic families of parabolic-like maps of degree 2.

Consider the family $Per_1(1)$. Note that for every $A \neq 0$, the map P_A has a parabolic fixed point of parabolic multiplicity 1, while the map $P_0 = z+1/z$ has a parabolic fixed point of parabolic multiplicity 2. Therefore, for every $A \neq 0$, a parabolic-like restriction of the map P_A has no attracting petals in its filled Julia set (for the definition of filled Julia set for a rational map see 2.5, and for the construction of a parabolic-like restriction of a map P_A see Prop.2.5.1), while a parabolic-like restriction of P_0 has exactly one attracting petal in its filled Julia set. On the other hand, all the maps of an analytic family of parabolic-like maps have the same number of attracting petals in their filled Julia set. Each (maximal) attracting petal requires a critical point in its boundary. Hence, there are exactly 2 possibilities for the number of attracting petals in the filled Julia set of an analytic family f_λ of parabolic-like maps of degree 2. Either for each $\lambda \in \Lambda$ the map f_λ has no attracting petals in K_{f_λ}, or for each $\lambda \in \Lambda$ the map f_λ has a exactly one attracting petal in K_{f_λ}.

In the second case, all the members of **f** are hybrid conjugate to the map $P_0 = z + 1/z$, hence the map

$$\chi : M_f \to M_1$$

is the constant map

$$\lambda \to 1,$$

(but this case is not really interesting).

On the other hand, in the first case, all the members of **f** have no petals in their filled Julia set. This means that there is no $\lambda \in \Lambda$ such that f_λ is hybrid conjugate to the map $P_0 = z + 1/z$, and finally the range of the map χ is not the whole of M_1, but it belongs to $M_1 \setminus \{1\}$. This is the case we are interested in.

3.2.2 Persistently and non persistently indifferent periodic points

Let $(R_\lambda)_{\lambda \in \Lambda}$ be an analytic family of rational maps. In the paper *'On the dynamics of rational maps'* (see [MSS]), Mañé, Sad and Sullivan introduce two partitions of Λ into a dense open set of parameters, for which the family is structurally stable, and its complement. In the first partition, structural stability is required on a neighborhood of the Julia set; in the second partition it is required on the Riemann sphere. In this section we study the first partition in our setting, since parabolic-like maps is a local concept. We will see that on the structurally stable set we can construct a holomorphic motion of the Julia set, and that the structurally stable set coincides with $\Lambda \setminus \partial M_f$.

Let $\mathbf{f} = (f_\lambda : U'_\lambda \to U_\lambda)$ be an analytic family of parabolic-like mappings. An indifferent periodic point z' for f_{λ_0}, is called *persistent* if for each neighborhood $V(z')$ of z' there exists a neighborhood $W(\lambda_0)$ of λ_0 such that, for every $\lambda \in W(\lambda_0)$ the map f_λ has in $V(z')$ an indifferent periodic point z'_λ of the same period and multiplier. Hence, if for some $\hat{\lambda} \in \Lambda$ all the periodic points of $f_{\hat{\lambda}}$ are hyperbolic, then, for all $\lambda \in \Lambda$ (since Λ is connected), f_λ does not have persistently indifferent periodic points (see [MSS]).

Let us define

- $I = \{\lambda \,|\, f_\lambda \text{ has in } \Omega'_\lambda \text{ a non persistently indifferent periodic point}\}$,
- $F = \overline{I}$,
- $R = \Lambda \setminus F$.

Note that:

1. for all $\lambda \in \Lambda$ the parabolic fixed point z_λ belongs to $\partial\Omega'_\lambda$ (and not to Ω'_λ);
2. the parabolic fixed point is persistent. Indeed, if $\mathbf{f} = (f_\lambda : U'_\lambda \to U_\lambda)$ is an analytic family of parabolic-like mappings and z_0 is the parabolic fixed point of f_{λ_0}, for each neighborhood $V(z_0)$ of z_0 there exists a neighborhood $W(\lambda_0)$ of λ_0 such that, for every $\lambda \in W(\lambda_0)$ the map f_λ has in $V(z_0)$ a parabolic fixed point of multiplier 1, by definition of analytic family of parabolic-like mappings.

Proposition 3.2.2. *Locally on R there exists a dynamic holomorphic motion of the Julia set, i.e. choosing $\lambda_0 \in R$ there exists a neighborhood $W(\lambda_0)$ of λ_0 and a map $\tau : W(\lambda_0) \times J_{\lambda_0} \to \mathbb{C}$ such that:*

1. $\forall z \in J_{\lambda_0},\ \tau_{\lambda_0}(z) = z$;
2. *τ is holomorphic in λ and injective in z;*
3. $\forall \lambda \in W(\lambda_0),\ f_\lambda \circ \tau_\lambda = \tau_\lambda \circ f_{\lambda_0}$.

Moreover, for all $\lambda \in W(\lambda_0)$ the map $\tau_\lambda : J_{\lambda_0} \to \mathbb{C}$ is quasiconformal.

Proof. The proof follows the one in [MSS], we give it here for completeness.

Let $\lambda_0 \in R$, and let $W(\lambda_0)$ be a neighborhood of λ isomorphic to a disk.

Claim 3.2.1. *For every repelling periodic point z_{λ_0} of f_{λ_0} there exists an analytic map*

$$z : W(\lambda_0) \to \overline{\mathbb{C}}$$

$$\lambda \to z(\lambda),$$

such that $z(\lambda)$ is a repelling periodic of f_λ of the same period as z_{λ_0}.

Proof. Let z_{λ_0} be a repelling periodic point for f_{λ_0} of period k. Hence it is a solution of the equation $\psi(\lambda_0, z) = f^k_{\lambda_0} - z = 0$, and since it is a repelling point, $\partial_z \psi(\lambda_0, z_{\lambda_0}) \neq 0$. Thus by the implicit function theorem there exists $W \times V(z_0)$ neighborhood of $(\lambda_0, z_{\lambda_0})$ such that $\forall \lambda \in W\ \exists! z_\lambda \in V(z_0) : f^k_\lambda(z_\lambda) = z_\lambda$, i.e., there exists a holomorphic function $z(\lambda) : W \to V(z_0)$ which associates to any λ the z_λ such that $f^k_\lambda(z_\lambda) = z_\lambda$. Let $\hat{\lambda} \in \partial W \cap \overline{W}(\lambda_0)$. Then $lim_{\lambda \to \hat{\lambda}} z(\lambda) = z(\hat{\lambda})$ is a repelling periodic point of $f_{\hat{\lambda}}$ of period k, since $W(\lambda_0) \subset R$. Then by the implicit function theorem there exists $\hat{W} \times V(z(\hat{\lambda}))$ neighborhood of $(\hat{\lambda}, z(\hat{\lambda}))$ such that $\forall \lambda \in \hat{W}\ \exists! \hat{z}_\lambda \in V(z(\hat{\lambda})) : f^k_\lambda(\hat{z}_\lambda) = \hat{z}_\lambda$. By uniqueness, $\forall \lambda \in W \cap \hat{W},\ z_\lambda = \hat{z}_\lambda$, hence we can extend z_λ to all of $W(\lambda_0)$. ☐

Call B_{λ_0} the set of the repelling periodic points of f_{λ_0}. Hence we obtain a holomorphic motion $\tau : W(\lambda_0) \times B_{\lambda_0} \to \mathbb{C}$ of the repelling periodic points of f_{λ_0}. Indeed:

1. $\tau_{\lambda_0} = z_0$, i.e. τ_{λ_0} is the identity on z_0,

2. $\forall \lambda \in W(\lambda_0)$ the map $\tau_\lambda(z_0)$ is injective,

3. the map $\tau_z(\lambda) = z_\lambda$ is holomorphic by construction.

Remark 3.2.1. *The condition $\forall \lambda \in W(\lambda_0)$ the map τ_λ is injective is trivially satisfied because $\lambda \in R$. Indeed, injectivity means that if there exists λ such that $\tau_\lambda(z_1) = \tau_\lambda(z_2)$, then $z_1 = z_2$. In other words, this means that the orbit $\tau_\lambda(z_1) = \tau(\lambda, z_1)$ will never cross the orbit $\tau_\lambda(z_2) = \tau(\lambda, z_2)$, when $z_1 \neq z_2$. The only case in which they can intersect is when two orbits $\tau_\lambda(z_1)$ and $\tau_\lambda(z_2)$ collapse in the same, i.e. when two hyperbolic periodic points collapse in the same parabolic one. Since we are in R, this cannot happen.*

Since the Julia set is the closure of repelling points, by the $\lambda-$Lemma we obtain a holomorphic motion of the Julia set

$$\tau : W(\lambda_0) \times J_{\lambda_0} \to \mathbb{C}.$$

This holomorphic motion is dynamic. Indeed, if z_0 is a repelling periodic point of period k for f_{λ_0}, by construction $z_\lambda = \tau_\lambda(z_0)$ is a repelling periodic point of period k for f_λ. Hence $\tau(\lambda, z)$ is a conjugacy between repelling periodic points and therefore by continuity it is a conjugacy between Julia sets. $\square$

Proposition 3.2.3. *The dynamic holomorphic motion* $\tau : W(\lambda_0) \times J_{\lambda_0} \to \mathbb{C}$ *constructed locally on* R *in Prop. 3.2.2 extends to a dynamic holomorphic motion*

$$\tau : W(\lambda_0) \times U(J_{\lambda_0}) \to \mathbb{C}$$

where $U(J_{\lambda_0})$ *is a neighborhood of the Julia set* J_{λ_0}.

For a proof we refer to [MSS] pg.$210-215$ (in the case **f** has Siegel disks or Herman rings see the proof in [S]).

Corollary 3.2.1. *Let* W *be a connected component of* R*. If* $\lambda_1, \lambda_2 \in W$*, then* $K_{\lambda_1}, K_{\lambda_2}$ *are quasiconformally homeomorphic. In particular, either* $W \subset M_f$ *or* $W \cap M_f = \emptyset$.

Proof. If $\lambda_1, \lambda_2 \in W$, where W is a connected component of R, then J_{λ_1} and J_{λ_2} are quasiconformally homeomorphic (since there is a local holomorphic motion of the Julia set). If K_{λ_1} and K_{λ_2} have interior, let $K_{i_1},\ K_{i_2}, 1 \le i \le n, \exists n \ge 1$ be the connected components of K_{λ_1} and K_{λ_2} respectively (K_{λ_1} and K_{λ_2} have the same number of connected components, since the dynamics on J_{λ_1} and J_{λ_2} are quasiconformally conjugate). For every $i, 1 \le i \le n$, let $G_{i_1},\ G_{i_2}$ be quasicircles in $U(J_{\lambda_1}) \cap \overset{\circ}{K}_{\lambda_{i_1}}$ and $U(J_{\lambda_2}) \cap \overset{\circ}{K}_{\lambda_{i_2}}$ respectively. Let $\phi_{i_1} : \overset{\circ}{K}_{i_1} \to \mathbb{D},\ \phi_{i_2} : \overset{\circ}{K}_{i_2} \to \mathbb{D}$ be Riemann maps and define $\mathbb{S}_{i_1} = \phi(G_{i_1})$ and $\mathbb{S}_{i_2} = \phi(G_{i_2})$. Then the homeomorphism $\varphi := \phi_{i_2} \circ \tau \circ \phi_{i_1}^{-1} : \mathbb{S}_{i_1} \to \mathbb{S}_{i_2}$ is quasisymmetric, hence it extends to a quasiconformal map $\Phi : \overline{\mathbb{D}}_{i_1} \to \overline{\mathbb{D}}_{i_2}$. Therefore, for every $i,\ 1 \le i \le n$ we can define a quasiconformal homeomorphism $\phi_{i_2}^{-1} \circ \Phi \circ \phi_{i_1} : K_{i_1} \to K_{i_2}$, and thus K_{λ_1} is quasiconformally homeomorphic to K_{λ_2}.

Finally, either $\lambda_1, \lambda_2 \in M_f$, or both $\lambda_1, \lambda_2 \notin M_f$, since there can not be a homeomorphism between a connected set and a disconnected one.

$\square$

Proposition 3.2.4. *(a) The interior of $M_f \subset R$*

(b) $R = \Lambda \setminus \partial M_f$

Proof. The proof follows the one in [DH]. We give it here for completeness. (a) Choose $\lambda_0 \in \overset{\circ}{M}_f$, and suppose f_{λ_0} has a non-persistent indifferent periodic point α_0 of period k and multiplicity n. Let $V(\alpha_0)$ be a round disk neighborhood of α_0 such that α_0 is the only periodic point of period k in $\overline{V}(\alpha_0)$. Let Λ_0 be a neighborhood of λ_0 in $\overset{\circ}{M}_f$ such that, for all $\lambda \in \Lambda_0$, f_λ has in $V(\alpha_0)$ n periodic points counted with multiplicity and λ_0 is the only parameter for which f_λ has in $V(\alpha_0)$ a degenerate periodic point of period k. Let $W(0)$ be a n-covering of Λ_0 branched at 0. Then there exists a branched covering $\lambda : W(0) \to \Lambda_0$, $t \to t^n + \lambda_0$, such that $\lambda(0) = \lambda_0$. Note that if α_0 is a simple indifferent periodic point, the map λ is the transalations by λ_0.

By the Implicit Function Theorem there exist W, V neighborhoods of 0, α_0 respectively ($W \subset W(0)$, and by taking a restriction of W, we can assume $V \subset V(\alpha_0)$) and a holomorphic map

$$\alpha : W \to V$$

$$t \to \alpha(t) = \alpha_{\lambda(t)},$$

such that $\alpha(0) = \alpha_0$, $f^k_{\lambda(t)}(\alpha(t)) = \alpha(t)$, and $(f^k_{\lambda(t)}(\alpha(t)))' = \rho(t)$ where $\rho : W \to \mathbb{C}^*$ is a non constant holomorphic function (non constant since the indifferent periodic point α_0 is non persistent, holomorphic because $f_{\lambda(t)}(z)$ is holomorphic in both λ and z, and the periodic cycle moves holomorphically). Again by the Implicit Function Theorem the critical point $c_{\lambda(t)} = c(t)$ moves holomorphically.

Let (t_n) be a sequence in W converging to 0, such that $|\rho(t_n)| < 1\ \forall n$. Then, for each n, $\alpha(t_n)$ is an attracting periodic point of period k for $f_{\lambda(t_n)}$. Hence the critical point belongs to the attracting basin of $\alpha(t_n)$ (and there exists i, $0 \le i \le k$ for which $f^i_{\lambda(t_n)}(c(t_n))$ belongs to the immediate basin of attraction of $\alpha(t_n)$). Therefore, for each n, we have:

$$f^{i+kp}_{\lambda(t_n)}(c(t_n)) \to \alpha(t_n) \text{ as } p \to \infty.$$

We can assume i independent of λ by choosing a subsequence. Let us define on W the sequence

$$F_p(t) = f^{i+kp}_{\lambda(t)}(c(t)).$$

Note that $\{F_p\}_{p\in\mathbb{N}}$ is a family of analytic maps (since f_λ are analytic) bounded on any compact subset of W (because $\lambda(t) \in M_f$ for every $t \in W$, and thus $F_p(t) \in K_{\lambda(t)}$). Hence it is a normal family. Let F_{p_n} be a subsequence converging to some function $h : W \to \mathbb{C}$. Then $h(t_n) = \alpha(t_n)$ for all n, and by the uniqueness of analytic continuation, $h = \alpha$ and for all $t \in W$, $F_p(t) \to \alpha(t)$.

Since $\lambda(0) = \lambda_0$ is a non persistent indifferent periodic point, in W there are points t^* such that $|\rho(t^*)| > 1$, thus $\alpha(t^*)$ is a repelling periodic point and it cannot attract the sequence $F_p(t^*)$. Thus $\mathring{M}_f \cap I = \emptyset$, and since $\mathring{M}_f$ is open, $\mathring{M}_f \cap F = \emptyset$ and finally $\mathring{M}_f \subset R$

(b)For the previous corollary, if W is a connected component of R, then $W \subset M_f$ or $W \cap M_f = \emptyset$. This implies that $R \cap \partial M_f = \emptyset$. Therefore $R \subset \Lambda \setminus \partial M_f$.

By (a) $\mathring{M}_f \subset R$, then we need to prove $(\Lambda \setminus M_f) \subset R$. For any $\lambda \in \Lambda$, since $d = 2$ the map f_λ has a unique critical point ω_λ. If $\lambda \in (\Lambda \setminus M_f)$ then $\omega_\lambda \notin K_\lambda$. Hence $\omega_\lambda \in (U'_\lambda \setminus K_\lambda)$, and any periodic point of f_λ which is not the parabolic fixed point is repelling. Therefore $(\Lambda \setminus M_f) \cap I = \emptyset$, and since $\Lambda \setminus M_f$ is open, $(\Lambda \setminus M_f) \subset R$.

□

3.3 Holomorphic motion of a fundamental annulus A_{λ_0} and Tubings

In chapter 2 we proved that a degree 2 parabolic-like map is hybrid conjugate to a member of the family $Per_1(1)$, by changing its external class into h_2, which is the external class of the family $Per_1(1)$. In other words we glued outside a degree 2 parabolic-like map f the map h_2. More precisely, we constructed a quasiconformal C^1 diffeomorphism $\widetilde{\psi}$ between a *fundamental annulus* A_f of the parabolic-like map and a *fundamental annulus* A of h_2. Then we defined on A_f an almost complex structure σ_1 by pulling back by $\widetilde{\psi}$ the standard structure σ_0. In order to obtain on U_f a bounded and invariant (under a map coinciding with f on Ω_f) almost complex structure σ we replaced the parabolic-like map with h_2 on Δ, and spread σ_1 by the dynamics of this new map $\tilde{f}$ (and kept the standard structure on K_f). Finally, by integrating σ we obtained a parabolic-like map hybrid conjugate to f and with external map h_2.

In this chapter we want to perform this surgery for an analytic family of parabolic-like maps, and we want to do it with some regularity with respect to the parameter. Hence we have to define a family of quasiconformal maps, depending holomorphically on the parameter, between a fundametal annulus of h_2 and a fundamental annulus of f_λ. In analogy with the polynomial-like setting we will call this family a *holomorphic Tubing*. Therefore we have to start by constructing a *fundamental annulus* for h_2 and for $(f_\lambda)_{\lambda\in\Lambda}$

In chapter 2 we already costructed a quasiconformal C^1-diffeomorphism $\widetilde{\psi}$ between a fundamental annulus of the parabolic-like map and a fundamental annulus of h_2. That construction shows that the fundamental annulus for h_2 depends on the parabolic-like map we start with. Therefore in this section we will first fix a $\lambda_0 \in \Lambda$, construct a fundamental annulus for h_2 and one for f_{λ_0}, and recall the quasiconformal C^1-diffeomorphism $\widetilde{\psi}$ between these fundamental annuli. Then we will derive fundamental annuli for f_λ from the fundamental annulus of f_{λ_0} by a *holomorphic motion*. Finally we will obtain a *holomorphic Tubing* by composing the inverse of $\widetilde{\psi}$ with the holomorphic motion.

Notation. *The term fundamental annulus is used here not in the sense of covering maps.*

A fundamental annulus A for h_2

The map $h_2(z) = \frac{z^2+1/3}{1+z^2/3}$ is the external map of the family $Per_1(1)$ (see Prop. 2.5.1). Let $h_2 : W' \to W$ (where $W = \{z : \exp(-\epsilon) < |z| < \exp(\epsilon)\},\ \epsilon > 0$, and $W' = h_2^{-1}(W)$) be a degree 2 covering. Choose $\lambda_0 \in \Lambda$. Let h_{λ_0} be an external map of f_{λ_0},z_0 be its parabolic fixed point and define $\gamma_{h_{\lambda_0}+} = \alpha_{\lambda_0}(\gamma_{\lambda_0+})$, $\gamma_{h_{\lambda_0}-} = \alpha_{\lambda_0}(\gamma_{\lambda_0-})$ (where $\alpha : \mathbb{C} \setminus K_\lambda \to \mathbb{C} \setminus \overline{\mathbb{D}}$ is the isomorphism which defines h_{λ_0}).

Let $\Xi_{h_f\pm}$ be repelling petals for the parabolic fixed point z_0 which intersect the unit circle and $\phi_\pm : \Xi_{h_f\pm} \to \mathbb{H}_-$ be Fatou coordinates for h_{λ_0} with axis tangent to the unit circle at the parabolic fixed point z_0. Let $\Xi_{h\pm}$ be repelling petals which intersect the unit circle for the parabolic fixed point $z = 1$ of h_2, and let $\widetilde{\phi}_\pm : \Xi_{h\pm} \to \mathbb{H}_-$ be Fatou coordinates for h_2 with axis tangent to the unit circle at 1. Define $\widetilde{\gamma}_+ = \widetilde{\phi}_+^{-1}(\phi_{h_{\lambda_0}+}(\gamma_{h_{\lambda_0}+}))$ and $\widetilde{\gamma}_- = \widetilde{\phi}_-^{-1}(\phi_{h_{\lambda_0}-}(\gamma_{h_{\lambda_0}-}))$.

Define $\widetilde{\Delta}_W = h_2(\Delta_W \cap \Delta'_W)$, $\widetilde{W} = \Omega_W \cup \widetilde{\gamma} \cup \widetilde{\Delta}_W$, $\widetilde{W'} = h_2^{-1}(\widetilde{W})$, $\widetilde{\Omega}'_W = \Omega'_W \cap \widetilde{W'}$, $\widetilde{\Delta}'_W = \Delta'_W \cap \widetilde{W'}$ and $Q_W = \Omega_W \setminus \widetilde{\Omega}'_W$ (see the proof of Theorem 2.4.5). We call *fundamental annulus for* h_2 the topological annulus $A = W \setminus (\overline{\widetilde{\Omega}'_W} \cup \mathbb{D})$.

A fundamental annulus A_{λ_0} for f_{λ_0} and a quasiconformal C^1 diffeomorphism $\widetilde{\Psi} : A \to A_{\lambda_0}$

Let ψ be a quasiconformal map between ∂U_{λ_0} and the outer boundary of W, such that $\psi(\gamma_{\lambda_0+}(1)) = \widetilde{\gamma}_+(1)$ and $\psi(\gamma_{\lambda_0-}(1)) = \widetilde{\gamma}_-(1)$. Let $\Phi_{\Delta_{\lambda_0}} : \Delta_{\lambda_0} \to \Delta_W$ be a quasiconformal C^1 diffeomorphism which extends to ψ on ∂U_{λ_0} and to $\widetilde{\phi}_{\pm}^{-1} \circ \phi_{h_{\lambda_0}\pm} \circ \alpha_{\lambda_0}$ on $\gamma_{\lambda_0 \pm}$ (see Claim 2.4.1 in the proof of Theorem 2.4.5). Define $\widetilde{\Delta}_{\lambda_0} = \Phi_{\Delta_{\lambda_0}}^{-1}(\widetilde{\Delta}_W)$, $\widetilde{\Delta'}_{\lambda_0} = \Phi_{\Delta_{\lambda_0}}^{-1}(\widetilde{\Delta}'_W)$, $\widetilde{U}_{\lambda_0} = (\Omega_{\lambda_0} \cup \gamma_{\lambda_0} \cup \widetilde{\Delta}_{\lambda_0}) \subset U_{\lambda_0}$. Consider

$$\widetilde{f}_{\lambda_0}(z) = \begin{cases} \Phi_{\Delta_{\lambda_0}}^{-1} \circ h_2 \circ \Phi_{\Delta_{\lambda_0}} & \text{on } \widetilde{\Delta'}_{\lambda_0} \\ f_{\lambda_0} & \text{on } \Omega'_{\lambda_0} \cup \gamma_{\lambda_0} \end{cases}$$

Define $\widetilde{U'}_{\lambda_0} = \widetilde{f}_{\lambda_0}^{-1}(\widetilde{U}_{\lambda_0})$, $Q_{\lambda_0} = \Omega_{\lambda_0} \setminus \overline{\widetilde{\Omega}'}_{\lambda_0}$, and the *fundamental annulus* $A_{\lambda_0} = U_{\lambda_0} \setminus \overline{\widetilde{\Omega}'}_{\lambda_0}$.

Let $\bar{\psi} : \partial\widetilde{U}_{\lambda_0} \to \partial(\widetilde{W} \cup \mathbb{D})$ be quasiconformal map coinciding with ψ on the outer boundary of Ω_{λ_0}, and let $\psi_1 : \partial\widetilde{U'}_{\lambda_0} \to \partial(\widetilde{W'} \cup \mathbb{D})$ be the lift of $\bar{\psi} \circ \widetilde{f}_{\lambda_0}$ to h_2 which preserves the dynamics on the dividing arcs. Let $\Phi_{Q_{\lambda_0}} : \overline{Q}_{\lambda_0} \to \overline{Q}_W$ be a quasiconformal C^1 diffeomorphism which coincides with $\bar{\psi}$ on $\partial\Omega_{\lambda_0}$, with ψ_1 on $\partial\widetilde{\Omega}_{\lambda_0}$ and with $\widetilde{\phi}_{\pm}^{-1} \circ \phi_{h_{\lambda_0}\pm} \circ \alpha_{\lambda_0}$ on $\gamma_{\lambda_0 \pm}$ (see the proof of Claim 4.2.2 in Theorem 2.4.5). Define a map $\widetilde{\psi} : A_{\lambda_0} \to A$ as follows :

$$\widetilde{\psi}(z) = \begin{cases} \widetilde{\phi}_{\pm}^{-1} \circ \phi_{h_{\lambda_0}\pm} \circ \alpha_{\lambda_0} & \text{on } \gamma_{\lambda_0 \pm} \\ \Phi_{\Delta_{\lambda_0}} & \text{on } \Delta_{\lambda_0} \\ \Phi_{Q_{\lambda_0}} & \text{on } Q_{\lambda_0} \end{cases}$$

This map is a quasiconformal C^1 diffeomorphism which extends continuously to the boundaries and quasiconformally to $\partial A_{\lambda_0} \setminus \{z_{\lambda_0}\}$ (where z_{λ_0} is the parabolic fixed point of f_{λ_0}). Therefore the map $\widetilde{\Psi} := \widetilde{\psi}^{-1} : A \to A_{\lambda_0}$ is a quasiconformal C^1 diffeomorphism which extends to a homeomorphism $\widetilde{\Psi} : \overline{A} \to \overline{A}_{\lambda_0}$ quasiconformal on $\overline{A} \setminus \{1\}$

Holomorphic motion of the fundamental annulus A_{λ_0}

Define for all $\lambda \in \Lambda$ the set $a_\lambda = U_\lambda \setminus \overline{\Omega'}_\lambda$. Then the set a_λ is a topological annulus. Define the map $\widetilde{\tau} : \Lambda \times \partial a_{\lambda_0} \to \partial a_\lambda$ as follows:

$$\widetilde{\tau}(z) = \begin{cases} \Phi_\lambda & \text{on } \gamma_{\lambda_0} \\ B_\lambda & \text{on } \partial U_{\lambda_0} \\ f_\lambda^{-1} \circ B_\lambda \circ f_{\lambda_0} & \text{on } \partial U'_{\lambda_0} \cap \partial\Omega'_{\lambda_0} \end{cases}$$

Let us show that $\widetilde{\tau}$ is a holomorphic motion with basepoint λ_0.

Indeed:

1. $\forall z_0 \in \gamma_{\lambda_0}$, $\widetilde{\tau}(z_0) = \Phi_{\lambda_0}(z_0) = z_0$ since Φ is a holomorphic motion, $\forall z_0 \in \partial U_{\lambda_0}$, $\widetilde{\tau}(z_0) = Id(z_0) = z_0$, and $\forall z_0 \in \partial U'_{\lambda_0} \cap \partial\Omega'_{\lambda_0}$, $\widetilde{\tau}(z_0) = f_{\lambda_0}^{-1} \circ Id \circ f_{\lambda_0} = z_0$;

2. the map $\widetilde{\tau}$ is injective in z, since Φ_λ and B_λ are holomorphic motions with disjoint images on $\partial a_{\lambda_0} \setminus \gamma_{\lambda_0 \pm}(\pm 1)$, and $f_\lambda : \partial U'_\lambda \to \partial U_\lambda$ is a degree d covering;

3. the map $\widetilde{\tau}$ is holomorphic in λ, since Φ_λ and B_λ are holomorphic motions, and the map f_λ depends holomorphically on λ.

Since $\Lambda \approx \mathbb{D}$, by the Slodkowski's theorem we can extend $\widetilde{\tau}$ to a holomorphic motion $\widetilde{\tau} : \Lambda \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. In particular we obtain a holomorphic motion of the set $\Lambda \times \widetilde{U}_{\lambda_0}$. For every $\lambda \in \Lambda$ define $\widetilde{U}_\lambda = \widetilde{\tau}(\widetilde{U}_{\lambda_0})$, and $\widetilde{\Delta'}_\lambda = \widetilde{\tau}(\widetilde{\Delta'}_{\lambda_0})$. Define for every $\lambda \in \Lambda$ the map $\widetilde{f}_\lambda$ as follows:

$$\widetilde{f}_\lambda(z) = \begin{cases} \widetilde{\tau} \circ \widetilde{\Psi} \circ h_2 \circ \widetilde{\Psi}^{-1} \circ \widetilde{\tau}^{-1} & \text{on } \widetilde{\Delta'}_\lambda \\ f_\lambda & \text{on } \Omega'_\lambda \cup \gamma_{f_\lambda} \end{cases}$$

and the set $\widetilde{U}'_\lambda = \widetilde{f}_\lambda^{-1}(\widetilde{U}_\lambda)$. Finally, define for all $\lambda \in \Lambda$ the set $A_\lambda = U_\lambda \setminus \overline{\widetilde{\Omega'}_\lambda}$. Then the set A_λ is a topological annulus, and we call it the *fundamental annulus of* f_λ. The holomorphic motion $\widetilde{\tau} : \Lambda \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ restricts to a holomorphic motion

$$\widehat{\tau} : \Lambda \times A_{\lambda_0} \to A_\lambda$$

which respects the dynamics. Note that, by construction, this holomorphic motion extends to the boundaries, and the extension respects the dynamics.

Holomorphic Tubings

Define $T := \widehat{\tau} \circ \widetilde{\Psi} : \Lambda \times A \to A_\lambda$. The map T is not a holomorphic motion, since $T_{\lambda_0} = \widetilde{\Psi} \neq Id$, but nevertheless it is quasiconformal in z for every fixed $\lambda \in \Lambda$ and holomorphic in λ for every fixed $z \in A$.

Definition 3.3.1. Let us denote by **holomorphic tubing** the map $T := \widehat{\tau} \circ \widetilde{\Psi} : \Lambda \times A \to A_\lambda$.

By construction, for every $\lambda \in \Lambda$, the map $T_\lambda^{-1} : A_\lambda \to A$ is a quasiconformal map which allows us to conjugate the map f_λ to a member of the family $Per_1(1)$. Indeed, for every $\lambda \in \Lambda$ we define on U_λ the Beltrami form μ_λ as follows:

$$\mu_\lambda(z) = \begin{cases} \mu_{\lambda,0} = \widehat{T_{\lambda *}}(\sigma_0) & \text{on } A_\lambda \\ \mu_{\lambda,n} = (\widetilde{f}_\lambda^n)^* \mu_{\lambda,0} & \text{on } (\widetilde{f}_\lambda)^{-n}(A_\lambda) \\ 0 & \text{on } K_\lambda \end{cases}$$

For every λ the map $\widehat{T}_\lambda$ is quasiconformal, then its inverse is quasiconformal, hence $||\mu_{\lambda,0}||_\infty \leq k < 1$ on every compact subset of Λ. On $\widetilde{\Omega}'_\lambda$ the Beltrami form $\mu_{\lambda,n}$ is obtained by spreading $\mu_{\lambda,0}$ by the dynamics of f_λ, which is holomorphic, while on Δ_λ the Beltrami form $\mu_{\lambda,n}$ is constant for all n (by construction of the map $\widetilde{f_\lambda}$). Hence the dilatation of $\mu_{\lambda,i}$ is constant. Therefore $||\mu_\lambda||_\infty = ||\mu_{\lambda,0}||_\infty$ which is bounded. By the measurable Riemann mapping theorem (see [Ah]) for every $\lambda \in \Lambda$ there exists a quasiconformal map $\phi_\lambda : U_\lambda \to \mathbb{D}$ such that $(\phi_\lambda)^*\mu_0 = \mu_\lambda$. Finally, for every $\lambda \in \Lambda$ the map $g_\lambda = \phi_\lambda \circ f_\lambda \circ \phi_\lambda^{-1}$ is the parabolic-like map hybrid conjugate to f_λ and holomorphically conjugate to a member of the family $Per_1(1)$.

Remark 3.3.1. *Note that for every $\lambda \in \Lambda$, the dilatation of the integrating map ϕ_λ is equal to the dilatation of the holomoprhic Tubing T_λ, and hence it is locally bounded.*

Lifting Tubings

By construction, the holomorphic motion $\widehat{\tau} : \Lambda \times A_{\lambda_0} \to A_\lambda$ extends to a holomorphic motion of the boundaries, and the quasiconformal C^1 diffeomorphism $\widetilde{\Psi} : A \to A_{\lambda_0}$ extends continuously to the boundaries and quasiconformally to $\overline{A} \setminus \{1\}$. Therefore, a holomorphic Tubing $T : \Lambda \times A \to A_\lambda$ extends to a holomorphic tubing $T : \Lambda \times \overline{A} \to \overline{A_\lambda}$ (note that the extension is just continuous, and quasiconformal on $\overline{A} \setminus \{1\}$), and the extension respects the dynamics.

Let us lift the Tubing T. Define $A_{\lambda,0} = \overline{\widetilde{U}_\lambda} \setminus \widetilde{\Omega}_\lambda$, $B_{\lambda,1} = \widetilde{f}_\lambda^{-1}(A_{\lambda,0})$, $A_0 = \overline{\widetilde{W}_\lambda} \setminus \widetilde{\Omega}_W$ and $B_1 = h_2^{-1}(A_0)$. Hence $\widetilde{f}_\lambda : B_{\lambda,1} \to A_{\lambda,0}$ and $h_2 : B_1 \to A_0$ are degree 2 covering maps, and, since by construction $T_\lambda(A_0) = A_{\lambda,0}$, we can lift the Tubing T_λ to $T_{\lambda,1} := \widetilde{f}_\lambda^{-1} \circ T_\lambda \circ h_2 : B_1 \to B_{\lambda,1}$ (such that $T_{\lambda,1} = T_\lambda$ on $B_1 \cap B_0$).

Define recursively $A_{\lambda,n} = B_{\lambda,n} \cap \widetilde{U}$, $B_{\lambda,n+1} = \widetilde{f}_\lambda^{-1}(A_{\lambda,n})$, $A_n = B_n \cap \widetilde{W}$ and $B_{n+1} = h_2^{-1}(A_n)$. Hence $\widetilde{f}_\lambda : B_{\lambda,n+1} \to A_{\lambda,n}$ and $h_2 : B_{n+1} \to A_n$ are degree 2 covering maps, and we can lift the Tubing to $T_{\lambda,n+1} := \widetilde{f}_\lambda^{-1} \circ T_{\lambda,n} \circ h_2 : B_{n+1} \to B_{\lambda,n+1}$ (such that $T_{\lambda,n+1} = T_{\lambda,n}$ on $B_{n+1} \cap B_n$).

In the case K_λ is connected, we can lift the Tubing T_λ to all of $W \setminus \overline{\mathbb{D}}$. If K_λ is not connected, the maximum domain we can lift the Tubing T_λ to is B_{n_0}, such that B_{λ,n_0} contains the critical value of f_λ. Note that the extension is still quasiconformal in z.

3.4 Properties of the map χ

By theorem 2.5.3 in chapter 2 if f is a parabolic-like map of degree $d = 2$, f is hybrid equivalent to a member of the family $Per_1(1)$, and if K_f is connected this member is unique. Therefore, if $\mathbf{f} = (f_\lambda : U'_\lambda \to U_\lambda)$ is an analytic family of parabolic-like maps of degree 2, the map

$$\chi : M_f \to M_1 \setminus \{1\}$$
$$\lambda \to B,$$

which associates to each $\lambda \in M_f$ the multiplier of the fixed point of the map P_A hybrid equivalent to f_λ is well defined (see 3.1). As we said, the aim of this chapter is to prove that the map χ extends to the whole of Λ, and the restriction to M_f is a branched covering of $M_1 \setminus \{1\}$. In this section, we will first extend the map χ to all of Λ (see 3.4.1), then prove that the map $\chi : \Lambda \to \mathbb{C}$ is continuous (see 3.4.2) and finally that it depends analytically on λ for $\lambda \in \mathring{M}_f$ (see 3.4.3).

3.4.1 Extending the map χ to all of Λ

By Tubings we can extend the map χ to the whole parameter space Λ. Since Tubings are not unique, the extension given here (which follows the one in [DH]) is not canonical, but it is anyway, given a Tubing, the 'natural' one.

Let T_λ be a holomorphic tubing for the analytic family of parabolic-like maps $\mathbf{f}$. Call c_λ the critical point of f_λ and let n be such that $f_\lambda^n(c_\lambda) \in A_\lambda$, $f_\lambda^{n-1}(c_\lambda) \notin A_\lambda$. Hence we can iterativaly lift the holomorphic tubing T_λ to $T_{\lambda,n-1} := \widetilde{f}_\lambda^{-1} \circ T_{\lambda,n-2} \circ h_2 = f_\lambda^{-(n-1)} \circ T_\lambda \circ h_2^{n-1} : B_{n-1} \to B_{\lambda,n-1}$ (where h_2^{n-1}, $\widetilde{f}_\lambda^{-(n-1)}$ are the branches which preserve the dynamics on the overlapping domains, see 3.3).

We can therefore extend the map χ to the whole of Λ by setting:

$$\chi : \Lambda \setminus M_f \to \mathbb{C} \setminus M_1$$
$$\lambda \to \Phi^{-1} \circ T_{\lambda,n-1}^{-1}(c_\lambda)$$

where $\Phi : \mathbb{C} \setminus M_1 \to \mathbb{C} \setminus \overline{\mathbb{D}}$ is the canonical isomorphism between the complement of M_1 and the complement of the unit disk. Since the maps $h_2 : B_{n-1} \to A_{n-2}$ and $\widetilde{f}_\lambda : B_{\lambda,n-1} \to A_{\lambda,n-2}$ are degree 2 coverings, the map Φ is an isomorphism, and the Tubing T_λ is a holomorphic tubing (and then quasiconformal in z) the map $\chi : \Lambda \setminus M_f \to \mathbb{C} \setminus M_1$ is quasiregular.

3.4.2 Continuity of the map χ

In this section we prove that the map $\chi : \Lambda \to \mathbb{C}$ is continuous. Since the map $\chi : \Lambda \setminus M_f \to \mathbb{C} \setminus M_1$ is quasiregular, we will start by proving that χ is continuous on $\mathring{M}_f$, and then we will prove continuity on the whole of Λ.

For every $\lambda \in M_f$ the parabolic-like map f_λ is hybrid conjugate to a unique member of the family $Per_1(1)$. This means that, if μ, μ' are two different Beltrami forms on U_λ obtained by spreading by the dynamics of $\widetilde{f}_\lambda$, $\widetilde{f}'_\lambda$ the pull back of the standard structure under two different quasiconformal maps $\psi : A_\lambda \to A$, and $\psi' : A_\lambda \to A$, then $P_{A(\lambda)} = \phi \circ \widetilde{f}_\lambda \circ \phi^{-1}$ and $P_{A'(\lambda)} = \phi' \circ \widetilde{f}'_\lambda \circ \phi'^{-1}$ (where $(\phi)^*\mu_0 = \mu$, $(\phi')^*\mu_0 = \mu'$) are in the same class $[P_A]$.

For this reason we are free to use a different Tubing $T'_\lambda = \widehat{\tau}' \circ \widetilde{\Psi} : A \to A_{\lambda_0}$ which defines a different almost complex structure μ'_λ on U_λ but yields to the same class hybrid conjugate to f_λ.

We will indeed define a different Tubing, since to prove continuity of the straightening map on $\mathring{M}_f$ we will need the Tubing to be a C^1-*diffeomorphism* in z. Therefore we start by constructing a *diffeomorphic motion* $\widehat{\tau}' : A_{\lambda_0} \times M_f \to A_\lambda$, i.e. a map no longer holomorphic in λ and quasiconformal in z but a C^1 diffeomorphism in z continuous in both (λ, z).

Diffeomorphic motion

Let $(\alpha_\lambda)_{\lambda \in \mathring{M}_f}$ be a family of Riemann maps $\alpha_\lambda : \mathbb{C} \setminus K_\lambda \to \mathbb{C} \setminus \overline{\mathbb{D}}$, normalized by $\alpha_\lambda(\infty) = \infty$ and $\alpha_\lambda(\gamma_\lambda(t)) \to 1$ as $t \to 0$. Since we can define locally on R a holomorphic motion of the Julia set (see 3.2.2), and $\mathring{M}_f \subset R$ (see 3.2.4), the family $(\alpha_\lambda)_{\lambda \in \mathring{M}_f}$ is continuous on λ. Let $(h_\lambda)_{\lambda \in \mathring{M}_f}$ be the associated family of external maps (see 2.3 in chapter 2), then $h_\lambda : W'_\lambda \to W_\lambda$ is a continuous family of holomorphic maps. In the rest of this subsection we will consider the parameter λ in the interior of M_f without further reference.

Define the dividing arcs $\gamma_{h_\lambda \pm} = \alpha_\lambda(\gamma_\lambda \pm)$, and note that the map α_λ extends to a homeomorphism $\alpha_\lambda : \gamma_\lambda \to \gamma_{h_\lambda}$ conjugating the dynamics of f_λ and h_λ. Define the set $A_{h_\lambda} = \alpha_\lambda(A_\lambda)$. Then the set A_{h_λ} is a topological annulus, and we call it the *fundamental annulus for* h_λ. We will construct a motion of the annulus A_{λ_0} by constructing a motion of the annulus $A_{h_{\lambda_0}}$.

The holomorphic motion $\widehat{\tau} : \Lambda \times A_{\lambda_0} \to A_\lambda$ extends by the λ-Lemma (see [MSS]) to a holomorphic motion of the boundaries $\widehat{\tau} : \Lambda \times \partial A_{\lambda_0} \to \partial A_\lambda$. Therefore, the family $\alpha_\lambda \circ \widehat{\tau} \circ \alpha_{\lambda_0}^{-1} : \Lambda \times \partial A_{h_{\lambda_0}} \to \partial A_{h_\lambda}$ is a family of homeomorphisms, (since α_λ extends to a homeomorphism conjugating the dynamics on the arcs), quasisymmetric on $\partial A_{h_{\lambda_0}} \setminus z_0$, (where z_0 is the parabolic fixed point of h_{λ_0}) and continuous in (λ, z) (see Fig.3.1).

Let us show that the family $\alpha_\lambda \circ \widehat{\tau} \circ \alpha_{\lambda_0}^{-1}$ is quasisymmetric on a neighborhood of the parabolic fixed point z_0. Let $\Xi_{h_{\lambda_0}+}$, $\Xi_{h_{\lambda_0}-}$, $\Xi_{h_\lambda+}$, and $\Xi_{h_\lambda-}$ be the repelling petals where $\gamma_{h_{\lambda_0}+}$, $\gamma_{h_{\lambda_0}-}$, $\gamma_{h_\lambda+}$, and $\gamma_{h_\lambda-}$, respectively reside, and let $\phi_{h_{\lambda_0}\pm} : \Xi_{h_{\lambda_0}\pm} \to \mathbb{H}_-$, and $\phi_{h_\lambda\pm} : \Xi_{h_\lambda\pm} \to \mathbb{H}_-$ be Fatou coordinates, normalized by mapping the unit circle to the negative real axis. Let $m_{\lambda+}, m_{\lambda-}$ be a sequence of real numbers continuous in λ, and set $\gamma_{s_\lambda+}(t) = \phi^{-1}_{h_\lambda+}(log_d(t) - m_{\lambda+}i),\ 0 \leq t \leq 1,\ \gamma_{s_\lambda-}(t) = \phi^{-1}_{h_\lambda-}(log_d(-t) + m_{\lambda-}i),\ -1 \leq t \leq 0$. Define the translations $T_{(\lambda_0,\lambda)+} = m_{\lambda_0+}i - m_{\lambda+}i$ and $T_{(\lambda_0,\lambda)-} = -m_{\lambda_0+}i + m_{\lambda+}i$. By Prop. 2.3.11,(3) there exist a quasisymmetric conjugacy $\delta_0 : \gamma_{h_{\lambda_0}} \to \gamma_{s_{\lambda_0}}$ between h_{λ_0} and itself and, for every λ, there exist quasisymmetric conjugacies $\delta_\lambda : \gamma_{h_\lambda} \to \gamma_{s_\lambda}$ between h_λ and itself. The proof of Prop. 2.3.11,(1) shows that for every λ the map $\phi^{-1}_{h_\lambda} \circ T_{(\lambda_0,\lambda)} \circ \phi_{h_{\lambda_0}} : \gamma_{s_{\lambda_0}} \to \gamma_{s_\lambda}$ is a quasisymmetric conjugaciy between h_{λ_0} and h_λ. Writing the map $\alpha_\lambda \circ \widehat{\tau} \circ \alpha^{-1}_{\lambda_0}|_{\gamma_{h_{\lambda_0}}} : \gamma_{h_{\lambda_0}} \to \gamma_{h_\lambda}$ as $\delta^{-1}_\lambda \circ \phi^{-1}_{h_\lambda} \circ T_{(\lambda_0,\lambda)} \circ \phi_{h_{\lambda_0}} \circ \delta_0$, is now clear that this map is quasisymmetric on a neighborhood of the parabolic fixed point z_0.

Consider the topological annulus A_{h_λ} as the union of two quasidisks: $Q_{h_\lambda} = \Omega_{h_\lambda} \setminus \overline{\widetilde{\Omega}'}_{h_\lambda}$ and Δ_{h_λ} (see Figure 3.1). The sets ∂Q_{h_λ} and $\partial\Delta_{h_\lambda}$ are quasicircles, since they are piecewise C^1 closed curves with non zero interior angles. Indeed, $\gamma_{h_\lambda+}$ and $\gamma_{h_\lambda-}$ are tangent to $\mathbb{S}^1$ at the parabolic fixed point (see the proof of 2.3.11), and we can assume the angles between $\gamma_{h_\lambda\pm}$ and $\partial(W_\lambda \cup \mathbb{D}),\ \partial(W'_\lambda \cup \mathbb{D})$ 'close to $\pi/2$'-in the sense that we can assume them to be positive and smaller then π- (we may take parabolic-like restrictions).

To obtain a diffeomorphic motion of the annulus $A_{h_{\lambda_0}}$ we construct diffeomorphic motions of the quasidisks $Q_{h_{\lambda_0}}$ and $\Delta_{h_{\lambda_0}}$ using the Douady-Earle extension. Let $\psi_{Q_\lambda} : Q_{h_\lambda} \to \mathbb{D},\ \lambda \in \mathring{M}_f$ be a family of Riemann maps depending continuously on λ, and let $\phi_{Q_\lambda} : \mathbb{D} \to Q_{h_\lambda}$ be the family of inverse maps. Then ϕ_{Q_λ} depends continuously on λ and extends continuously to the boundaries, and since ∂Q_{h_λ} is a quasicircle the family $\phi_{Q_\lambda} : \mathbb{S}^1 \to \partial Q_{h_\lambda}$ is quasisymmetric in z and continuous in (λ, z). Hence the family of quasisymmetric homeomorphisms $\varphi_{Q_\lambda} := \phi^{-1}_{Q_\lambda} \circ \alpha_\lambda \circ \widehat{\tau} \circ \alpha^{-1}_{\lambda_0} \circ \phi_{Q_{\lambda_0}} : \mathbb{S}^1 \to \mathbb{S}^1$ continuous in (λ, z), extends (see [DE]) to a family of quasiconformal maps $\Phi_{Q_\lambda} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, which are C^1 diffeomorphism on $\mathbb{D}$, continuous in (λ, z). Then $\widehat{\Psi}_{Q_\lambda} := \phi_{Q_\lambda} \circ \Phi_{Q_\lambda} \circ \psi_{Q_{\lambda_0}} : Q_{h_{\lambda_0}} \to Q_{h_\lambda}$ is a family of quasiconformal maps which are C^1 diffeomorphisms, depending continuously on (λ, z) (see Fig.3.1).

On the other hand, let $\psi_{\Delta h_\lambda} : \Delta_{h_\lambda} \to \mathbb{D}$ be a family of Riemann maps depending continuously on λ, and let $\phi_{\Delta h_\lambda} : \mathbb{D} \to \Delta_{h_\lambda}$ be the family of inverse maps. Then $\phi_{\Delta h_\lambda}$ depends continuously on λ, and it extends continuously to the boundary. Moreover, since $\partial\Delta_{h_\lambda}$ is a quasicircle, the restriction $\phi_{\Delta h_\lambda} : \mathbb{S}^1 \to \partial\Delta_{h_\lambda}$ is quasisymmetric.

Define the family of homeomorphisms $\varphi_{\Delta h_\lambda} := \phi^{-1}_{\Delta h_\lambda} \circ \alpha_\lambda \circ \widehat{\tau} \circ \alpha^{-1}_{\lambda_0} \circ \phi_{\Delta h_{\lambda_0}} : \mathbb{S}^1 \to \mathbb{S}^1$ continuous in (λ, z). How we saw before, the map $\alpha_\lambda \circ \widehat{\tau} \circ \alpha^{-1}_{\lambda_0}$ is a quasisymmetric homeomorphism, hence the map $\varphi_{\Delta h_\lambda} : \mathbb{S}^1 \to \mathbb{S}^1$ is

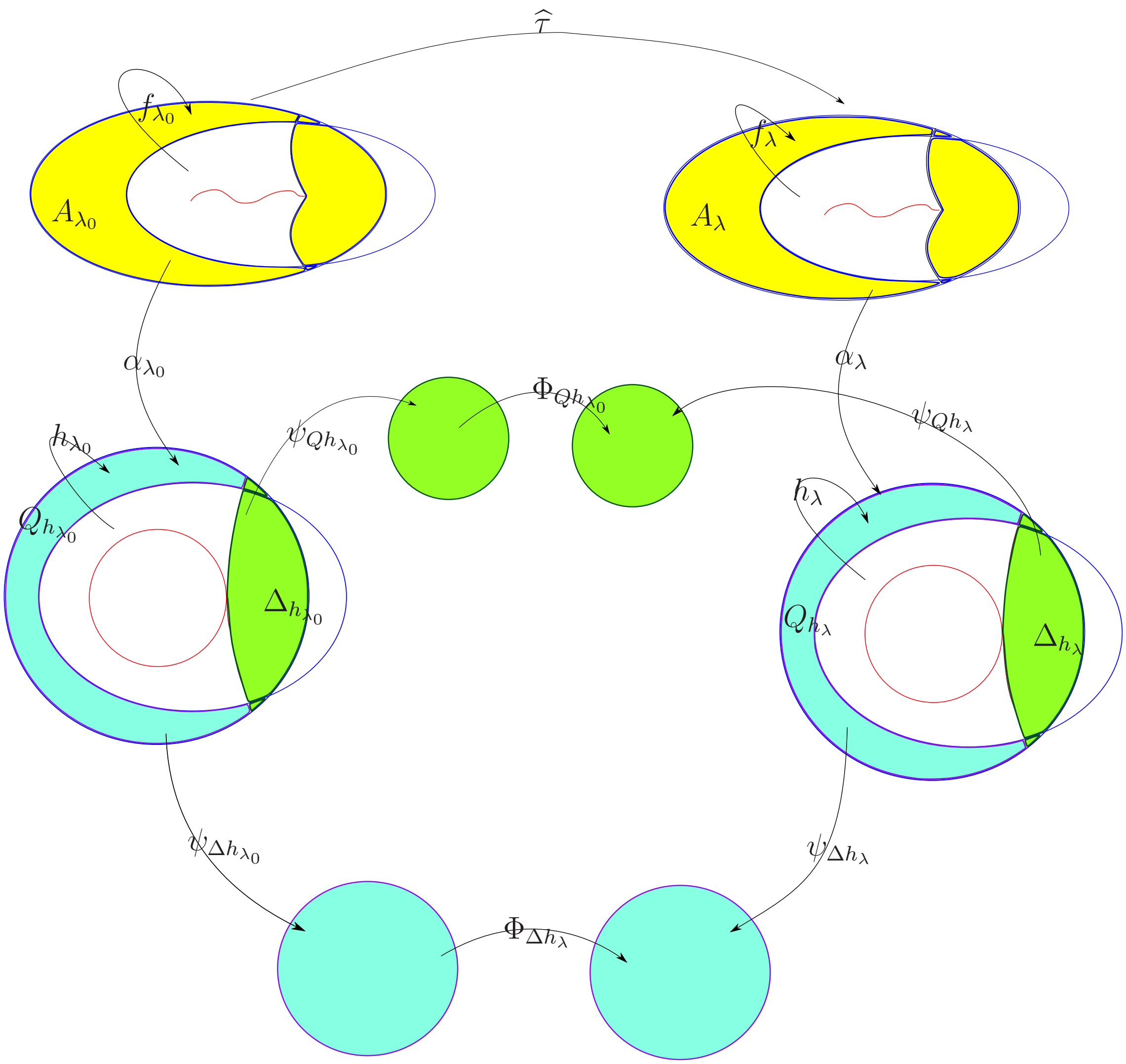


Figure 3.1: Construction of the diffeomorphic motion $\widehat{\tau'} : A_{\lambda_0} \times \mathring{M}_f \to A_\lambda$.

a quasisymmetric homeomorphism. Therefore the family $\varphi_{\Delta h_\lambda}$ extends by the Douady-Earle extension (see [DE]) to a family of quasiconformal maps $\Phi_{\Delta h_\lambda} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, real-analytic diffeomorphisms on $\mathbb{D}$, continuous in (λ, z).

Therefore, the family $\widehat{\Psi}_{\Delta_\lambda} := \phi_{\Delta_\lambda} \circ \Phi_{\Delta h_\lambda} \circ \psi_{\Delta_{\lambda_0}} : \Delta_{h_{\lambda_0}} \to \Delta_{h_\lambda}$ is a continuous (in both (λ, z)) family of quasiconformal maps which are C^1 diffeomorphisms, and which extends to $\alpha_\lambda \circ \Phi_\lambda \circ \alpha_{\lambda_0}^{-1}$ on $\gamma_{h_\lambda +}$ and $\gamma_{h_\lambda -}$.

Hence we can define a diffeomorphic motion $\widehat{\tau'} : A_{\lambda_0} \times \mathring{M}_f \to A_\lambda$ as follows:

$$\widehat{\tau'}(z) = \begin{cases} \alpha_\lambda^{-1} \circ \widehat{\Psi}_{Q_\lambda} \circ \alpha_{\lambda_0} & \text{on } Q_{\lambda_0} \\ \alpha_\lambda^{-1} \circ \widehat{\Psi}_{\Delta h_\lambda} \circ \alpha_{\lambda_0} & \text{on } \Delta_{\lambda_0} \\ \Phi_\lambda & \text{on } \partial Q_{\lambda_0} \cap \partial \Delta_{\lambda_0} = \gamma_{\lambda_0 +}[1/d, 1] \cup \gamma_{\lambda_0 -}[-1/d, -1] \end{cases}$$

where $\Phi : \Lambda \times \gamma_{\lambda_0} \to \mathbb{C}$ is the holomorphic motion of the dividing arcs (see 3.2.1). The family $\widehat{\tau'} : A_{\lambda_0} \times \mathring{M}_f \to A_\lambda$ is a family of quasiconformal maps which are C^1 diffeomorphisms, and which are continuous as a function of (λ, z).

We can now define a tubing which is quasiconformal and a C^1-diffeomorphism in z, and continuous in (λ, z).

Definition 3.4.1. Let us call **diffeomorphic tubing** the map $\widehat{T} := \widehat{\tau'} \circ \widetilde{\Psi} : \mathring{M}_f \times A \to A_\lambda$, where $\widetilde{\Psi} : A \to A_{\lambda_0}$ is the quasiconformal C^1 diffeomorphism constructed in 3.3.

Continuity of χ on $\mathring{M}_f$

Proposition 3.4.2. *On the open set $\mathring{M}_f$ both ϕ_λ and P_{A_λ} depend continuously on λ.*

Proof. The proof follows the one in [DH]. We write it here for completeness. Let $U \subset \mathbb{C}$ be compact, (μ_n) be a sequence of Beltrami forms on U and μ be another Beltrami form on U, then if:

1. $\exists m < 1 \ : \ ||\mu||_\infty \leq m$ and $||\mu_n||_\infty \leq m \ \forall n$,

2. $\mu_n \xrightarrow{L_1} \mu$,

the family of integrating maps ϕ_λ converges to ϕ uniformly on $\mathbb{C}$ (see [Hu], pg.154).

Since $||\mu_n||_\infty \leq m \ \forall n$ on any compact subset of Λ (see 3.3), the continuity of the straightening map (and thus of P_{A_λ}) follows by proving that

$$\mu_\lambda \stackrel{L_1}{\to} \mu_{\lambda_0} \text{ as } \lambda \to \lambda_0.$$

Define

$$\hat{\mu}_{\lambda,n}(z) = \begin{cases} \mu_{\lambda,i}(z) & \text{on } A_{\lambda,i} \text{ for } i \leq n \\ 0 & \text{on } U_{\lambda,n} = \widetilde{U}_\lambda \setminus \bigcup_i^{n-1} A_{\lambda,i} \end{cases}$$

Then $\mu_\lambda = lim_{n\to\infty} \hat{\mu}_{\lambda,n}$ pointwise. Since $|\mu_\lambda - \mu_{\lambda_0}|_{L^1} \leq |\mu_\lambda - \hat{\mu}_{\lambda,n}|_{L^1} + |\hat{\mu}_{\lambda,n} - \hat{\mu}_{\lambda_0,n}|_{L^1} + |\hat{\mu}_{\lambda_0,n} - \mu_{\lambda_0}|_{L^1}$, in order to prove $\mu_\lambda \stackrel{L_1}{\to} \mu_{\lambda_0}$ we need to prove that:

$$(a)\ \hat{\mu}_{\lambda,n} \xrightarrow{L_1} \mu_\lambda \text{ as } n \to \infty$$

$$(b)\ \hat{\mu}_{\lambda,n} \xrightarrow{L_1} \hat{\mu}_{\lambda_0,n} \text{ as } \lambda \to \lambda_0$$

$$(c)\ \hat{\mu}_{\lambda_0,n} \xrightarrow{L_1} \mu_{\lambda_0} \text{ as } n \to \infty$$

Clearly $(a) \Rightarrow (c)$, hence we have to prove (a) and (b). Let us start by proving (b).

(b) On Δ_λ the beltrami forms $\hat{\mu}_{\lambda,n}$ and $\hat{\mu}_{\lambda,0}$ coincide (by definition of $\widetilde{f_\lambda}$), and on Ω_λ the pull back is done by f_λ, which depends holomorphically on λ. Hence to show that for each n, $\hat{\mu}_{\lambda,n}$ depends continuously on λ in the L^1 norm, it is enough to show that $\hat{\mu}_{\lambda,0}$ depends continuously on λ in the L^1 norm, i.e.

$$\int |\hat{\mu}_{\lambda,0} - \hat{\mu}_{\lambda_0,0}| \xrightarrow{\lambda\to\lambda_0} 0.$$

Since:

$$\widehat{T}_\lambda : (A, \mu_0) \to (A_\lambda, \hat{\mu}_{\lambda,0})$$

we can compute

$$\hat{\mu}_{\lambda,0}(z) = (\widehat{T}_\lambda^{-1})^*(\mu_0)(z) = \frac{\partial\bar{z}\widehat{T}_\lambda^{-1}(z)}{\partial z\widehat{T}_\lambda^{-1}(z)}.$$

Since the diffeomorphic tubing $\widehat{T}_\lambda$ is a family of quasiconformal maps which are C^1-diffeomorphism in z and continuous in (λ, z), the family of derivatives $\widehat{T}'_\lambda$ and their inverse $(\widehat{T}_\lambda^{-1})'$ is continuous in (λ, z). Therefore $\partial\bar{z}\widehat{T}_\lambda^{-1}$ and $\partial z\widehat{T}_\lambda^{-1}$ are continuous in (λ, z), and thus $\hat{\mu}_{\lambda,0}$ depends continuously in (λ, z). Finally, since $\hat{\mu}_{\lambda,0}$ is continuous and bounded, it depends continuously in λ in the L^1 norm. Therefore $\hat{\mu}_\lambda$ depends continuously in λ in the L^1 norm.

(a) The fact that $\hat{\mu}_{\lambda,n} \xrightarrow{L_1} \mu_\lambda$ as $n \to \infty$ follows from the fact that the area of $U_{\lambda,n} \setminus K_\lambda$ tends to zero *uniformly* on every compact subset of R.

Indeed $\hat{\mu}_{\lambda,n}$ and μ_λ are different just on $U_{\lambda,n} \setminus K_\lambda$, hence

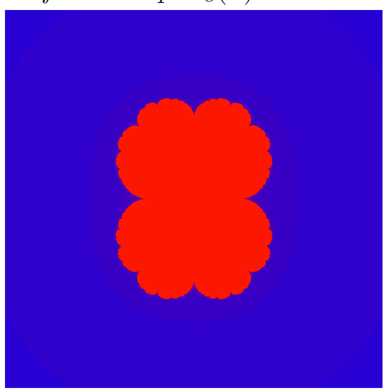
$$|\mu_\lambda - \hat{\mu}_{\lambda,n}|_{L_1} < \sup_{z\in(U_{\lambda,n}\setminus K_\lambda)} |\mu_\lambda(z) - \hat{\mu}_{\lambda,n}(z)| * area(U_{\lambda,n} \setminus K_\lambda).$$

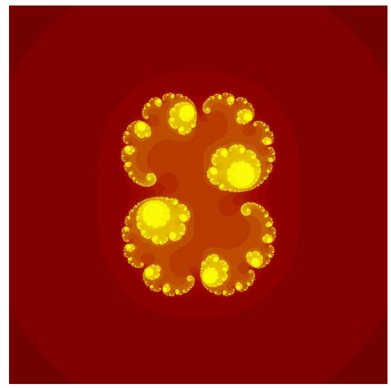
Since $\hat{\mu}_{\lambda,n} = 0$ on $U_{\lambda,n}$, and $\sup_z |\mu_\lambda| = ||\mu_\lambda||_\infty = ||\mu_{\lambda,0}||_\infty < 1$, we obtain the following bound:

$$|\mu_\lambda - \hat{\mu}_{\lambda,n}|_{L_1} < 1 * area(U_{\lambda,n} \setminus K_\lambda).$$

Therefore, $Area(U_{\lambda,n} \setminus K_\lambda) \xrightarrow{n\to\infty} 0$ locally uniformly on R implies $\hat{\mu}_{\lambda,n} \xrightarrow{L_1} \mu_\lambda$.

Remark 3.4.1. *The area of $U_{\lambda,n} \setminus K_\lambda$ does not tend to zero on any subset of Λ which intersects F. Indeed for every value of λ for which f_λ has a non persistent parabolic fixed point, the area of $U_{\lambda,n}$ still depends continuously on λ, but the area of K_λ is discontinuous. See the pictures below: the pictures on the left shows the filled Julia set of the map $P_{1/4}(z) = z^2 - 1/4$, which has a non persistent parabolic fixed point, and the picture on the right shows the filled Julia set of the map $P_c(z) = z^2 + c$ with $c = 0.285 + 0.01i$.*





Choose $\lambda_0 \in \mathring{M}_f$, let $W(\lambda_0)$ be a neighborhood of λ_0 in $\mathring{M}_f$ and consider a dynamic holomorphic motion

$$\tau(\lambda, z) : W(\lambda_0) \times U(J_{\lambda_0}) \to \mathbb{C}$$

$$(\lambda, z) \to z_\lambda$$

extension to a neighborhood $U(J_{\lambda_0})$ of J_{λ_0} of the dynamic holomorphic motion of the Julia set constructed locally on R in 3.2.2 (see 3.2.3). Define $U(J_\lambda) = \tau_\lambda(U(J_{\lambda_0}))$, $B_\lambda = U(J_\lambda) \cup K_\lambda$ and $B'_\lambda = f_\lambda^{-1}(B_\lambda)$. Then, for every $\lambda \in W(\lambda_0)$, $(f_\lambda, B'_\lambda, B_\lambda, \gamma_\lambda)$) is a parabolic-like restriction of $(f_\lambda, U'_\lambda, U_\lambda, \gamma_\lambda)$. Set $V_{\lambda,0} = B_\lambda \setminus \widetilde{\Omega}'_{B_\lambda}$, $V_{\lambda,n} = \widetilde{f}_\lambda^{-n}(V_{\lambda,0}) \cap B_\lambda$, $B_{\lambda,n} = B_\lambda \setminus \bigcup_{i=0}^{n-1} V_{\lambda,n}$.

There exists a neighborhood $W(\lambda_0)'$ of λ_0 with compact closure in $W(\lambda_0)$ and $p \in \mathbb{N}$ such that $U_{\lambda,p} \subset B_\lambda$ for all $\lambda \in W(\lambda_0)'$. We then obtain $U_{\lambda,p+n} \subset B_{\lambda,n}$.

Let us define $m_n(\lambda) = area(B_{\lambda,n} \setminus K_\lambda)$. Clearly $area(B_{\lambda,n} \setminus K_\lambda) \xrightarrow{u} 0$ implies $area(U_{\lambda,n} \setminus K_\lambda) \xrightarrow{u} 0$.

Since $U(J_\lambda) = \tau_\lambda(U(J_{\lambda_0}))$, $B_\lambda = U(J_\lambda) \cup K_\lambda$ and the holomorphic motion $\tau(\lambda, z) : W(\lambda_0) \times U(J_{\lambda_0}) \to \mathbb{C}$ is dynamic (hence $\tau_\lambda(V_{\lambda_0,n}) = V_{\lambda,n}$), we can write

$$m_n(\lambda) = \int_{B_{\lambda_0,n} \setminus K_{\lambda_0}} |Jac(\tau_\lambda)| dxdy.$$

Clearly $m_n \to 0$ pointwise. Set $[D\tau(z)] : T_z U_{\lambda_0} \to T_{\tau_\lambda(z)} U_\lambda$, $[D\tau(z)](u) = \frac{\partial \tau_\lambda(z)}{\partial z}(u) + \frac{\partial \tau_\lambda(z)}{\partial \bar{z}}(\bar{u})$, then $||D\tau_\lambda|| = \sup_{||z||=1} ||D\tau_\lambda(z)|| = |\frac{\partial \tau_\lambda}{\partial z}| + |\frac{\partial \tau_\lambda}{\partial \bar{z}}|$ and

$$Jac\tau_\lambda \leq ||D\tau_\lambda||^2 \leq KJac\tau_\lambda$$

where $K = \frac{1+|\mu_\lambda|}{1-|\mu_\lambda|} > 1$ and $\mu_\lambda = (\tau_\lambda)^* \mu_0$. Since τ_λ is holomorphic in λ, the sequence

$$n_n(\lambda) = \int_{B_{\lambda_0,n} \setminus K_{\lambda_0}} ||D\tau_\lambda||^2 dxdy,$$

is subharmonic. Since

$$m_n \leq n_n \leq Km_n,$$

we have that

$$\frac{1}{K} n_n \leq m_n \leq n_n.$$

Since $m_n \to 0$ pointwise, then $n_n \to 0$ pointwise. The sequence $n_n \to 0$ decreases, hence it is uniformly bounded on any compact set; and thus it converges in L^1_{loc} [Hö]. Since the limit function is constant, the sequence $n_n \to 0$ converges to zero uniformly on any compact subset of $W(\lambda_0)'$, and thus $m_n(\lambda) \to 0$ uniformly on any compact subset of $W(\lambda_0)'$.

Therefore on $\mathring{M}_f$ the straightening map ϕ_λ converges uniformly to ϕ_{λ_0} as $\lambda \to \lambda_0$, which implies that $P_{A_\lambda} := \phi_\lambda \circ \widetilde{f}_\lambda \circ \phi_\lambda^{-1}$ is continuous in λ on $\mathring{M}_f$.

□

Continuity of χ on Λ

The proofs of the statements in this subsection follow their analogous in the polynomial-like setting (see [DH]). We wrote them here for completeness.

Proposition 3.4.3. *Suppose $A_1, A_2 \in \mathbb{C}$, with $B_1 = 1 - (A_1)^2 \in \partial M_1$. If the maps P_{A_1} and P_{A_2} are quasiconformally conjugate, then $(A_1)^2 = (A_2)^2$.*

Proof. Let (P_1, U', U, γ_1) and (P_2, V', V, γ_2) be parabolic-like restrictions of P_{A_1} and P_{A_2} respectively, and let $\varphi : U \to V$ be a hybrid equivalence between them. If K_{P_1} is of measure zero (for the definition of filled Julia set for the members of the family $Per_1(1)$ see 2.5 in chapter 2), then ϕ is a hybrid conjugacy and the result follows from Prop. 2.5.2 in chapter 2.

Let K_{P_1} be not of measure zero. Define on $\widehat{\mathbb{C}}$ the following Beltrami form:

$$\widetilde{\mu}(z) := \begin{cases} (\phi)^* \mu_0 & \text{on } K_{P_1} \\ 0 & \text{on } \widehat{\mathbb{C}} \setminus K_{P_1} \end{cases}$$

Since ϕ is quasiconformal, $||\widetilde{\mu}||_\infty = k < 1$. Therefore for $|t| < 1/k$ we can define on $\widehat{\mathbb{C}}$ the family of Beltrami form $\mu_t = \widetilde{\mu}t$, and $||\mu_t||_\infty < 1$. The family μ_t depends holomorphically on t. Let

$$\Phi_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$$

be the family of quasiconformal maps such that $(\Phi_t)^*\mu_0 = \mu_t$, $\Phi_t(\infty) = \infty$, $\Phi_t(-1) = -1$ and $\Phi_t(0) = 0$. Then the family Φ_t depends holomorphically on t, $\Phi_1 = \phi$ and $\Phi_0 = Id$. The family of holomorphic maps $F_t = \Phi_t \circ P_{A_1} \circ \Phi_t^{-1}$ has the form $F_t(z) = z + 1/z + A(t)$ (since it is a family of quadratic rational maps with a parabolic fixed point at $z = \infty$ with preimage at $z = 0$ and a critical point at $z = -1$) and it depends holomorphically on t. Hence $\alpha : t \to B(t) = 1 - A^2(t)$ is a holomorphic map, with $\alpha(0) = B_1 \in \partial M_1$. Since $\alpha(t)$ is holomorphic, it is either an open or constant map. If $\alpha(t)$ is open, since $\alpha(0) \in \partial M_1 \subset M_1$, there exists a neighborhood W of 0 such that $\alpha(W) \subset M_1$. Since $\alpha(0) \in \partial M_1$, it is impossible. Hence the map $\alpha(t)$ is constant, and $\alpha(t) = B_1$, $\forall t$. In particular, for $t = 1$, we have $\alpha(1) = B_1$, and $F_1 = P_{A_1}$.

Finally the map $\phi \circ \Phi_1^{-1}$ is a quasiconformal conjugacy between P_{A_1} and P_{A_2}, with $(\phi \circ \Phi_1^{-1})^*\mu_0 = \mu_0$ on K_{P_1}, and hence hybrid. Therefore, by Prop. 2.5.2 in chapter 2, $(A_1)^2 = (A_2)^2$. □

Lemma 3.4.1. *Choose $\lambda_0 \in \Lambda$ and let (λ_n) be a sequence in Λ converging to λ_0. Then there exists a subsequence $(\lambda_k^*) = (\lambda_{k_n})$ such that the maps $P_{A_k^*}$ converge to a map $\widetilde{P_A}$ and such that the $\phi_{\lambda_k^*}$ converge uniformly on every compact subset of U_{λ_0} to a quasi-conformal equivalence $\widetilde{\phi}$ between f_{λ_0} and $\widetilde{P_A}$.*

Proof. Choose $\lambda_0 \in \Lambda$ and let (λ_n) be a sequence in Λ converging to λ_0. Let ϕ_{λ_n} be a family of hybrid conjugacies between f_{λ_n} and P_{A_n}. The maos ϕ_{λ_n} are quasiconformal with locally bounded dilatation (see 3.3 and Remark 3.3.1), hence they form an equicontinuous family (see [A] pg.49, or [Hu] pg. 129).

Since the ϕ_{λ_n} are equicontinuous, there exists a subsequence $\phi_{\lambda_k^*}$ which converges to some quasiconformal limit map $\widetilde{\phi}$ when $\lambda \to \lambda_0$. Hence:

$$f_{\lambda_n} \overset{\lambda\to\lambda_0}{\longrightarrow} f_{\lambda_0},$$

$$\phi_{\lambda_k^*} \overset{\lambda\to\lambda_0}{\longrightarrow} \widetilde{\phi}.$$

Therefore

$$P_{A_k^*} \overset{\lambda\to\lambda_0}{\longrightarrow} \widetilde{P_A},$$

where $\widetilde{P_A} = \widetilde{\phi} \circ f_{\lambda_0} \circ \widetilde{\phi}^{-1}$. □

Remark 3.4.2. *Note that the limit $\widetilde{\phi}$ of a subsequence $\phi_{\lambda_k^*}$ of hybrid conjugacies between the maps $f_{\lambda_k^*}$ and $P_{A_k^*}$ is just a quasiconformal conjugacy between the limit maps f_{λ_0} and $\widetilde{P}$. This is because $\overline{\partial}\phi_{\lambda_k^*} = 0$ on a measure zero set does not imply $\overline{\partial}\widetilde{\phi} = 0$ on a set with positive measure, and when $\lambda_n \to \lambda_0$ with $\lambda_n \notin M_f$ and $\lambda_0 \in \partial M_f$, the filled Julia sets of the maps belonging to the subsequences $f_{\lambda_k^*}$ and $P_{A_k^*}$ are without interior, while the filled Julia set of limit maps f_{λ_0} and $\widetilde{P}$ may have interior.*

Proposition 3.4.4. *The map $\chi : \Lambda \to \mathbb{C}$ is continuous.*

Proof. By Prop.3.4.2 the map χ is continuous on $\lambda \setminus \partial M_f$. Therefore, we need to prove that for any sequence $\lambda_n \in \Lambda$ converging to a point $\lambda_0 \in \partial M_f$, we can choose a subsequence λ_{n^*} such that $B_{n^*} = \chi(\lambda_n^*)$ converges to $B_0 = \chi(\lambda_0) \in \partial M_1$. Let us start by proving that $B_0 \in \partial M_1$.

Let λ_m be a sequence in ∂M_f converging to λ_0. By Lemma 3.4.1 there exists a subsequence λ_{m^*} such that $P_{A_{m^*}}$ converges to a $P_{A_{\widehat{m}}}$ quasiconformally equivalent to f_{λ_0}. For all m the map f_{λ_m} has an indifferent periodic point, hence P_{A_m} has an indifferent periodic point, thus $B_m \in \partial M_1$ and finally the limit $B_{\widehat{m}}$ belongs to ∂M_1. The map f_{λ_0} is hybrid conjugate to P_{A_0} and quasiconformally conjugate to $P_{A_{\widehat{m}}}$. Since $P_{A_{\widehat{m}}}$ is quasiconformally equivalent to P_{A_0} and $B_{\widehat{m}} \in \partial M_1$, by Prop.3.4.3, $P_{A_{\widehat{m}}}$ and P_{A_0} are in the same class. Hence $B_0 = \chi(\lambda_0)$ belongs to ∂M_1.

Now let $(\lambda_n) \in \Lambda$ be a sequence converging to λ_0. By the previous Lemma, there exists a subsequence (λ_{n^*}) such that:

$$\phi_{n^*} \overset{\lambda \to \lambda_0}{\longrightarrow} \widetilde{\phi},$$

and $\widetilde{\phi}$ is a quasiconformal conjugacy between f_{λ_0} and $P_{\widetilde{A}}$. Therefore f_{λ_0} is quasiconformally conjugate to both $P_{\widetilde{A}}$ and P_{A_0}. Hence by Prop. 3.4.3, $P_{\widetilde{A}}$ and P_{A_0} are in the same class of $Per_1(1)$. Finally, for every sequence $\lambda_n \in \Lambda$ converging to a point $\lambda_0 \in \partial M_f$, there exists a subsequence λ_{n^*} such that $B_{n^*} \to B_0 = \chi(\lambda_0) \in \partial M_1$, and hence the map χ is continuous. □

3.4.3 Analicity of χ on the interior of M_f

In this section we prove that the map $\chi : \Lambda \to \mathbb{C}$ depends analytically on λ for $\lambda \in \mathring{M}_f$ (Corollary 3.4.1), and that for all $B \in M_1 \setminus \{1\}$, $\chi^{-1}(B)$ is a complex analytic subset of M_f (Corollary 3.4.2).

Proposition 3.4.5. *Let $f = (f_\lambda : U'_\lambda \to U_\lambda)$ and $g = (g_\iota : V'_\iota \to V_\iota)$ be analytic families of parabolic-like maps of degree 2 parametrized respectively by Λ, $\Lambda \approx \mathbb{D}$ and I, $I \approx \mathbb{D}$. Let W_1 be a connected component $\mathring{M}_f$, W_2 a* *connected component of $\mathring{M}_g$. Then the set $\Gamma \subset W_1 \times W_2$ of those (λ, ι) for which f_λ and g_ι are hybrid equivalent is a complex-analytic subset of $W_1 \times W_2$.*

Proof. The proof follows the one in [DH], with the differences given by the geometry of our setting.
Choose $\iota_0 \in W_2$. and let $T_g : I \times A \to A_\iota$ be a holomorphic tubing of $(g_\iota)_{i \in I}$ (see 3.3). This defines a dividing arc $\widetilde{\gamma}$ and a fundamental annulus A for h_2 (see 3.3). Let us assume for all $\iota \in W_2$, $g_\iota^{-1}(\widetilde{\Delta}_\iota) \subset \Delta_\iota$ (in other case, take an analytic family of parabolic-like restriction of the family g_ι for which the assumption holds). Choose $\lambda_0 \in W_1$ and let Λ' be a neighborhood of λ_0 in W_1. In order to construct a holomorphic tubing for $(f_\lambda)_{\lambda \in \Lambda'}$ which *respects the fundamental annulus A for h_2* we first need to replace the dividing arc γ_{λ_0} with dividing arcs isotopic to it and such that the map $\phi_{h_{\lambda_0}} \circ \phi_h$ (where $\phi_{h_{\lambda_0}}$, ϕ_h are repelling Fatou coordinates for the external map h_{λ_0} of f_{λ_0} and h respectively) is a *quasisymmetric conjugacy* between $\alpha_{\lambda_0}(\gamma_{\lambda_0})$ and $\widetilde{\gamma}$. Moreover, we need that, for every $\lambda \in \Lambda'$, $\gamma_\lambda := \phi_\lambda^{-1} \circ \phi_h(\widetilde{\gamma})$ is a *dividing arc* for f_λ isotopic to the original. For this aim take, if necessary, a parabolic-like restriction of f_{λ_0} such that, for all $\lambda \in \Lambda'$, $U_{\lambda_0} \subseteq U_\lambda$, and there exists U_+, U_- neighborhoods of $\gamma_{\lambda_0}(1)$, $\gamma_{\lambda_0}(-1)$ respectively in ∂U_{λ_0} such that: for all $\lambda \in \Lambda'$, $\gamma_{\lambda\pm} \cap \partial U_{\lambda_0} \in U_\pm$ and $\phi_{\lambda\pm}(U_\pm)$ crosses $\phi_{\lambda_0 \pm} \circ (\alpha_{\lambda_0 \pm})^{-1} \circ (\phi_{h_{\lambda_0}\pm})^{-1} \circ \phi_{h\pm}(\widetilde{\gamma}_\pm)$ once and without horizontal slopes. Then, redefine the dividing arcs as $\gamma_\lambda := \phi_\lambda^{-1} \circ \phi_h(\widetilde{\gamma})$. This redefines on Λ' the holomorphic motion Φ_λ of the dividing arcs as $\Phi_\lambda(\gamma_{\lambda_0}) = \phi_\lambda^{-1} \circ \phi_{\lambda_0}(\gamma_{\lambda_0})$. For all $\lambda \in \Lambda'$, $(f_\lambda, U'_\lambda, U_{\lambda_0}, \gamma_\lambda)$ is a parabolic-like restriction of $(f_\lambda, U'_\lambda, U_\lambda, \gamma_\lambda)$ (note that the dividing arcs are isotopic but they do not coincide), and that $(f_\lambda)_{\lambda \in \Lambda'}$ (where $f_\lambda : U'_\lambda \to U_{\lambda_0}$) is an analytic (sub)family of parabolic like maps (note that the boundaries still move holomorphically).

Let $\Psi_{\lambda_0} : A \to A_{\lambda_0}$ be a quasiconformal C^1 diffeomorphism whose restriction $\Psi_{\lambda_0} : \widetilde{\gamma} \to \gamma_{\lambda_0}$ conjugates dynamics (the costruction of Ψ_{λ_0} is given by 3.3, the only difference is that the map which extends the quasiconformal C^1 diffeomorphism Ψ_{λ_0} on γ_λ here is $\phi_h^{-1} \circ \phi_\lambda(\gamma_\lambda)$). Define the holomorphic motion $\widehat{\tau} : \Lambda' \times \partial(U_{\lambda_0} \setminus \Omega'_{\lambda_0}) \to \partial(U_{\lambda_0} \setminus \Omega'_\lambda)$ as follows:

$$\widehat{\tau}_\lambda(z) := \begin{cases} Id & \text{on } U_{\lambda_0} \\ \Phi_\lambda & \text{on } \gamma_{\lambda_0} \\ f_\lambda^{-1} \circ f_{\lambda_0} & \text{on } \partial U'_{\lambda_0} \cap \partial \Omega_\lambda \end{cases}$$

where f_λ^{-1} is the branch which preserves the dynamics on the dividing arcs. Let $\overline{\tau} : \Lambda' \times A_{\lambda_0} \to A_\lambda$ be the restriction to the fundamental annulus A_{λ_0} of the extension (given by the Slodkowski theorem) to $\widehat{\mathbb{C}}$ of the holomorphic motion $\widehat{\tau}$. Therefore, $T_f := \overline{\tau} \circ \Psi_{\lambda_0} : \Lambda' \times A \to A_\lambda$ is a holomorphic tubing for $(f_\lambda)_{\lambda\in\Lambda'}$ which respects the fundamental annulus A for h_2.

Define for any $(\lambda \times \iota) \in \Lambda' \times W_2$ the map (see Figure 3.2):

$$\delta_{(\lambda,\iota)} := T_g \circ T_f^{-1} : \Lambda' \times W_2 \times A_\lambda \to A_\iota,$$

and define for any $\iota \in W_2$ the map:

$$\widetilde{\delta}_{(\iota)} := \delta_{\lambda,\iota} \circ \overline{\tau}_\lambda = T_g \circ \Psi_{\lambda_0}^{-1} : W_2 \times A_{\lambda_0} \to A_\iota$$

In order to prove that the set Γ of those (λ, ι) for which f_λ and g_ι are hybrid equivalent is a complex-analytic subset of $W_1 \times W_2$ we will now prove that:

1. For every $(\lambda, \iota) \in \Lambda' \times W_2$, the map $\delta_{(\lambda,\iota)}$ defines an almost complex structure on U_{λ_0} which depends *holomorphically* on (λ, ι);
2. the set of (λ, ι) for which the map $\delta_{(\lambda,\iota)} : A_\lambda \to A_\iota$ extends to a holomorphic map $\alpha : U_{\lambda_0} \to U_\iota$ which conjugates f_λ and g_ι equals Γ;
3. the set of (λ, ι) for which the map $\delta_{(\lambda,\iota)} : A_\lambda \to A_\iota$ extends to a holomorphic map $\alpha : U_{\lambda_0} \to U_\iota$ is a complex analytic subset of $W_1 \times W_2$.

Remark 3.4.3. *By costruction, for every $\lambda \in \Lambda'$ the range of the parabolic-like restriction of f_λ is U_{λ_0}. The fundamental annulus of f_λ is still dependent on λ, since it is $A_\lambda = U_{\lambda_0} \setminus \overline{\widetilde{\Omega}'}_\lambda$ (see 3.3).*

(1) For every $\lambda \in \Lambda' \setminus \lambda_0$ define on U_{λ_0} the following family of Beltrami forms:

$$\nu_{(\lambda,\iota)}(z) := \begin{cases} \nu_{\lambda,\iota,0} = (\delta_{(\lambda,\iota)})^* \mu_0 & \text{on } A_\lambda \\ (\widetilde{f}^n_{(\lambda,\iota)})^* \nu_{\lambda,\iota,0} & \text{on } A_{\lambda,n} \\ 0 & \text{on } K_\lambda \end{cases}$$

where in this case the map $\widetilde{f}_{(\lambda,\iota)}$ which spreads the Beltrami forms $\nu_{\lambda,\iota,0}$ and defines the sets $A_{\lambda,n}$ (following 3.3) depends on both (λ, ι), and it is defined as follows:

$$\widetilde{f}_{(\lambda,\iota)}(z) = \begin{cases} \delta^{-1}_{(\lambda,\iota)} \circ g_\iota \circ \delta_{(\lambda,\iota)} & \text{on } \delta^{-1}(g_\iota^{-1}(\widetilde{\Delta}_\iota)) \\ f_\lambda & \text{on } \widetilde{\Omega}'_\lambda \end{cases}$$

For λ_0 define on U_{λ_0} the following family of Beltrami forms:

$$\widetilde{\nu}_\iota(z) := \begin{cases} \widetilde{\nu}_{\iota,0} = \widetilde{\delta}^*_{(\iota)}(\mu_0) & \text{on } A_{\lambda_0} \\ (\widetilde{f}^n_{(\lambda_0,\iota)})^* \nu_{\iota,0} & \text{on } A_{\lambda_0,n} \\ 0 & \text{on } K_{\lambda_0} \end{cases}$$

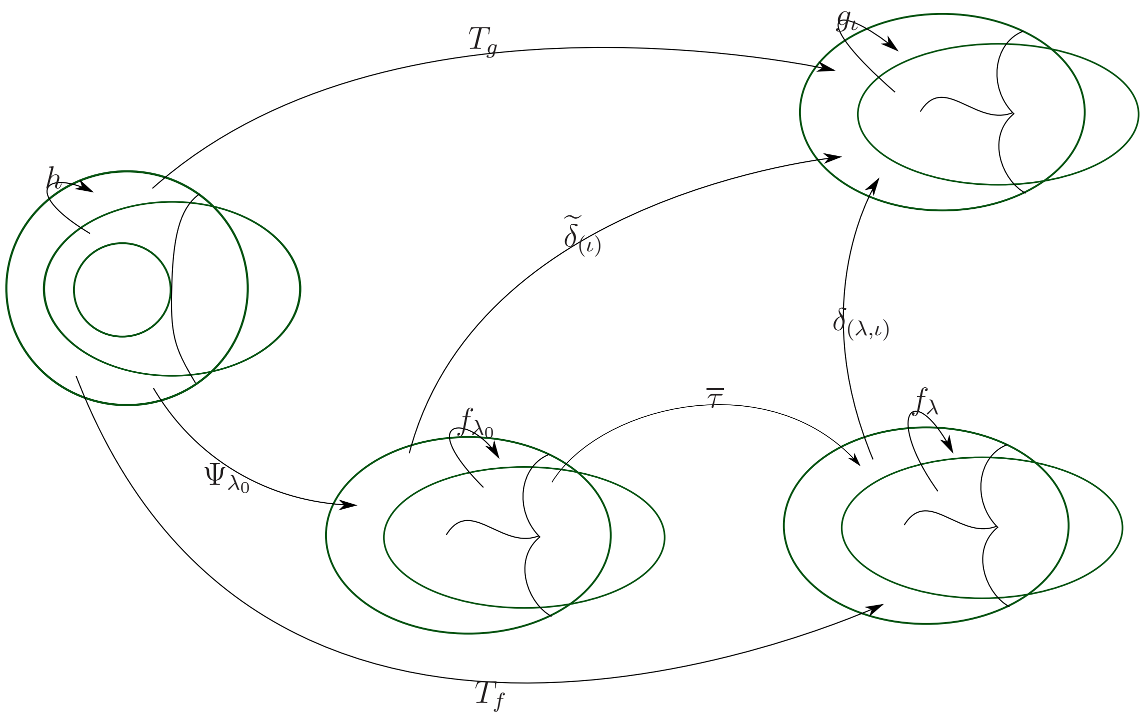


Figure 3.2: Construction of the maps $\delta_{(\lambda,\iota)} := T_g \circ T_f^{-1} : \Lambda' \times W_2 \times A_\lambda \to A_\iota$ and $\widetilde{\delta}_{(\iota)} :=:= \delta_{\lambda,\iota} \circ \overline{\tau}_\lambda = T_g \circ \Psi_{\lambda_0}^{-1} : W_2 \times A_{\lambda_0} \to A_\iota$.

(where $\widetilde{f}_{(\lambda_0,\iota)}$ and $A_{\lambda_0,n}$ are as above). Let us show that for every $z \in U_{\lambda_0}$ the map

$$\widetilde{\nu}_\iota(z) : I \longrightarrow L^\infty(U_{\lambda_0})$$

$$\iota \to (z \to \widetilde{\nu}_{(\iota)}(z))$$

is complex analytic in ι. Indeed, $\widetilde{\nu}_{\iota,0} = \widetilde{\delta}_\iota^*(\mu_0) = (T_{g_\iota} \circ \Psi_{\lambda_0}^{-1})^*(\mu_0) = (\Psi_{\lambda_0})_*(T_{g_\iota}^* \mu_0)$ is complex analytic in ι because T_g is a holomorphic tubing and Ψ_{λ_0} does not depend on ι. The Beltrami form $\widetilde{\nu}_{\iota,0}$ is spread on $\widetilde{\Omega}'_{\lambda_0}$ by the dynamics of f_{λ_0} (which does not depends on λ nor on ι), and on Δ_{λ_0} it is constant, hence for every $z \in U_{\lambda_0}$ the family $\widetilde{\nu}_\iota$ still depends holomorphically on ι.

By the Measurable Riemann Mapping theorem with parameters, there exist charts $\widetilde{\theta}_\iota : W_2 \times U_{\lambda_0} \to \mathbb{C}$ depending analytically on ι which integrate the Beltrami forms $\widetilde{\nu}_\iota$. On the other hand, there exist charts $\theta_{\lambda,\iota}$ which integrate the Beltrami forms $\nu_{\lambda,\iota}$, and by construction the following diagram commutes:

$$\begin{tikzcd}
\Lambda' \times W_2 \times A_{\lambda_0} \arrow[r, "(Id, \overline{\tau}_\lambda)"] \arrow[d, "(Id\ \widetilde{\theta}_\iota)"] & \Lambda' \times W_2 \times A_{\lambda_0} \arrow[d, "\theta_{\lambda,\iota}"] \\
\Lambda' \times W_2 \times \mathbb{C} \arrow[r, "(Id,\ Id)"] & \Lambda' \times W_2 \times \mathbb{C}
\end{tikzcd} \tag{3.1}$$

The fact that the previous diagram commutes implies that the following diagram commutes:

$$\begin{tikzcd}
\Lambda' \times W_2 \times A_{\lambda_0} \arrow[r, "(Id, \widetilde{\delta}_\iota)"] \arrow[d, "(Id, \widetilde{\theta}_\iota)"] & \Lambda' \times W_2 \times A_\iota \\
\Lambda' \times W_2 \times \mathbb{C} & \Lambda' \times W_2 \times A_{\lambda_0} \arrow[l, "\theta_{\lambda,\iota}"'] \arrow[u, "\delta_{\lambda,\iota}"']
\end{tikzcd} \tag{3.2}$$

hence $\widetilde{\delta}_\iota \circ \widetilde{\theta}_\iota^{-1} = \delta_{\lambda,\iota} \circ \theta_{\lambda,\iota}^{-1}$. This finally means that, since $\widetilde{\delta}_\iota \circ \widetilde{\theta}_\iota^{-1}$ depends holomorphically on the parameter ι, the map $\delta_{\lambda,\iota} \circ \theta_{\lambda,\iota}^{-1}$ depends holomorphically on the parameters (λ, ι), even if $\delta_{\lambda,\iota} \circ \theta_{\lambda,\iota}^{-1}$ depends a priori on (λ, ι) while $\widetilde{\delta}_\iota \circ \widetilde{\theta}_\iota^{-1}$ depends only on ι.

Let us now return to integrating the family of Beltrami forms $\nu_{\lambda,\iota}$. Next Lemma says that, if $(\lambda, \iota) \in \Gamma$ there exists an integrating map $\alpha_{(\lambda,\iota)}$ which conjugates f_λ to g_ι. That is, if $(\lambda, \iota) \in \Gamma$, there exists a map $\alpha_{(\lambda,\iota)} : U_\lambda \to \mathbb{C}$ such that $\alpha^*_{(\lambda,\iota)}(\mu_0) = \nu_{\lambda,\iota}$, extending $\delta_{(\lambda,\iota)}$ and conjugating f_λ and g_ι. This proves point (2).

Lemma 3.4.3 states that the set of (λ, ι) such that the map $\delta_{(\lambda,\iota)}$ extends to a map $\alpha_{(\lambda,\iota)} : U_\lambda \to V_\iota$ holomorphic with respect to θ (i.e. the set Γ) is a complex analytic submanifold. This proves point (3).

Lemma 3.4.2. *Recall that* $\Gamma := \{(\lambda, \iota) \in W_1 \times W_2 |\ f_\lambda$ *and* g_ι *are hybrid equivalent*$\}$. *For any* $(\lambda, \iota) \in \Lambda' \times W_2$ *the following conditions are equivalent:*

1. $(\lambda, \iota) \in \Gamma$,

2. *there exists an isomorphism*

$$\alpha = \alpha_{(\lambda,\iota)} : U_\lambda \to \mathbb{C}$$

$$(U_\lambda, \nu_{(\lambda,\iota)}) \to (V_\iota, \mu_0)$$

extending $\delta_{(\lambda,\iota)}$ *and conjugating* f_λ *and* g_ι,

3. *there exists a map* $\alpha : U_\lambda \to \mathbb{C}$ *holomorphic with respect to* $\nu_{(\lambda,\iota)}$ *and extending* $\delta_{(\lambda,\iota)}$.

Proof. To see that 2 implies 1 it is enough to remark that α is a conjugacy between f_λ and g_ι conformal with respect to $\nu_{(\lambda,\iota)}$, and thus f_λ and g_ι are hybrid equivalent. To see that 2 implies 3 note that an isomorphism with respect to $\nu_{(\lambda,\iota)}$ is a holomorphic map with respect to $\nu_{(\lambda,\iota)}$, and for all $\iota \in I$, $V_\iota \in \mathbb{C}$.

Let us show that 1 imples 2. Let β be a hybrid equivalence between f_λ and g_ι. Define the map $\alpha : U_\lambda \to V_\iota$ as follows:

$$\alpha(z) := \begin{cases} \delta_{(\lambda,\iota)} & \text{on } A_\lambda \\ \widetilde{g}_\iota^{-n} \circ \delta_{(\lambda,\iota)} \circ \widetilde{f}_\lambda^{n} & \text{on } A_{\lambda,n} \\ \beta & \text{on } K_\lambda \end{cases}$$

where the maps $\widetilde{g}_\iota$, $\widetilde{f}_\lambda$ are as in 3.3 and the sets $A_{\lambda,n}$ are constructed in 3.3. Then α is a hybrid conjugacy between f_λ and g_ι which is holomorphic with respect to $\nu_{(\lambda,\iota)}$ by construction (since the Beltrami form ν_λ is constant on Δ_λ, and the map β is hybrid). Since $\Lambda' \in M_f$, the proof of Prop. 2.4.4 in chapter 2 shows that the map α is quasiconformal. Hence α is an isomorphism with respect to $\nu_{(\lambda,\iota)}$ conjugating f_λ and g_ι, and it extends $\delta_{(\lambda,\iota)}$ by construction.

To show that 3 implies 2 we need to prove that the map $\alpha : U_\lambda \to \mathbb{C}$ is an isomorphism, i.e. that it has degree 1. To count the number of preimages under the map α it is enough to calculate the winding number of the image by α of a loop around a point belonging to U_λ. Since α is a holomorphic extension of $\delta_{(\lambda,\iota)}$, this is the winding number of the image by $\delta_{(\lambda,\iota)}$ of a loop around a point in A_λ, which is 1 since $\delta_{(\lambda,\iota)} = T_g \circ T_f^{-1}$. □

Lemma 3.4.3. *The set of (λ,ι) such that the map $\delta : A_\lambda \to A_\iota$ extends to a map $\alpha_{\lambda,\iota} : U_\lambda \to V_\iota$ holomorphic with respect to $\theta_{(\lambda,\iota)}$, is a complex analytic subset of $\Lambda' \times W_2$.*

Proof. The set of (λ,ι) such that the map $\delta : A_\lambda \to A_\iota$ extends to a map $\alpha_{\lambda,\iota} : U_\lambda \to V_\iota$ holomorphic with respect to $\theta_{(\lambda,\iota)}$, is the set of (λ,ι) such that the map $h_{(\lambda,\iota)} := \delta_{\lambda,\iota} \circ \theta_{\lambda,\iota}^{-1} : \theta_{\lambda,\iota}(A_\lambda) \to A_\iota$ extends to a holomorphic map $\alpha_{\lambda,\iota} \circ \theta_{\lambda,\iota}^{-1} : \theta_{\lambda,\iota}(U_\lambda) \to V_\iota$.

Chose $\iota_0 \in W_2$, and let I' be a neighborhood of ι_0 in W_2. Let $D_1 \subset\subset D_2$ be C^1 Jordan domains in $\theta_{\lambda,\iota}(A_\lambda)$ such that $\overline{D_2} \setminus D_1 \subset h_{(\lambda,\iota)}^{-1}(A_\iota)$ for all $(\lambda,\iota) \in \Lambda' \times I'$. Let γ_2 be the anticlockwise oriented Jordan curve which bounds D_2 and let γ_1 be the anticlockwise oriented Jordan curve which bounds D_1. Define $F(z) = \frac{1}{2\pi i} \oint_{\gamma_2} \frac{h_{(\lambda,\iota)}(w)}{w-z} dw$, and $G(z) = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{h_{(\lambda,\iota)}(w)}{w-z} dw$. Hence, by the Cauchy integral formula, on $D_2 \setminus \overline{D}_1$, $h_{(\lambda,\iota)}(z) = F(z) - G(z)$.

It is clear that, if $G \equiv 0$, $h_{(\lambda,\iota)}(z) = F(z)$ on $D_2 \setminus \overline{D}_1$, hence $h_{(\lambda,\iota)} = F$ and therefore $h_{(\lambda,\iota)}$ extends holomorphically (and the extension coincides with F)

on $\theta_{\lambda,\iota}(U_\lambda)$. On the other hand, if $h_{(\lambda,\iota)}$ extends holomorphically on $\theta_{\lambda,\iota}(U_\lambda)$, by the Cauchy integral formula, on D_2, $h_{(\lambda,\iota)} = \frac{1}{2\pi i}\oint_{\gamma_2} \frac{h_{(\lambda,\iota)}(w)}{w-z}dw = F(z)$, hence $h_{(\lambda,\iota)} = F$ and thus $G \equiv 0$. Therefore, to prove that $h_{(\lambda,\iota)}$ extends holomorphically on $\theta_{\lambda,\iota}(U_\lambda)$, we need to prove that $G \equiv 0$.

We have:

$$G(z) = \frac{1}{2\pi i}\oint_{\gamma_1} \frac{h_{(\lambda,\iota)}(w)}{w-z}dw = \frac{1}{2\pi i}\oint_{\gamma_1} h_{(\lambda,\iota)}(w)\cdot\frac{1}{w-z}dw =$$

since $(-\frac{1}{z}\sum_{n=0}^{\infty}(\frac{w}{z})^n) = \frac{1}{w-z}$

$$= \frac{1}{2\pi i}\oint_{\gamma_1} h_{(\lambda,\iota)}(w)\cdot(-\frac{1}{z}\sum_{n=0}^{\infty}(\frac{w}{z})^n))dw = \frac{1}{2\pi i}\oint_{\gamma_1} h_{(\lambda,\iota)}(w)\cdot(-\sum_{n=0}^{\infty}\frac{w^n}{z^{n+1}})dw =$$

$$= -\frac{1}{2\pi i}(\sum_{n=0}^{\infty}\frac{1}{z^{n+1}})\oint_{\gamma_1} h_{(\lambda,\iota)}(w)\cdot w^n dw.$$

Set

$$b_n = -\frac{1}{2\pi i}\oint_{\gamma_1} h_{(\lambda,\iota)}(w)\cdot w^n dw,$$

we obtain

$$G(z) = \sum_{n=0}^{\infty}\frac{b_n}{z^{n+1}},$$

hence $G \equiv 0$ if and only if $\forall n \geq 0,\ \ b_n = 0$. Since all the b_n are holomorphic maps in (λ,ι), this is a complex analytic set. Since the set of (λ,ι) for which $h_{(\lambda,\iota)}$ extends holomorphically to $\theta(U_{\lambda_0})$ is the set of (λ,ι) such that the map $\delta : A_\lambda \to A_\iota$ extends to a map $\alpha_{\lambda,\iota} : U_\lambda \to V_\iota$ holomorphic with respect to $\theta_{(\lambda,\iota)}$, we obtain that this is a complex analytic set. □

Since this set is Γ, we obtain that Γ is a complex analytic subset of $W_1 \times W_2$. □

Corollary 3.4.1. *The map $\chi_\lambda : \lambda \to B$ depends analytically on λ for $\lambda \in \mathring{M}_f$.*

Proof. Let us apply the previous Proposition to $f_\lambda,\ \lambda \in M_f$ and $P_A,\ B = 1 - A^2 \in M_1 \setminus \{1\}$. Since the graph of χ_λ is the set of (λ, B) for which f_λ is hybrid equivalent to P_A, this is a complex analytic set. Since on $\mathring{M}_f$ the map χ_λ is continuous and f_λ does not have no persistent indifferent periodic points, the map χ_λ is analytic on $\mathring{M}_f$. □

Corollary 3.4.2. *If $\widehat{B} \in M_1 \setminus \{1\}$, then $\chi^{-1}(\widehat{B})$ is an analytic subset of M_f.*

Proof. Let $\widehat{B} = 1 - \widehat{A}^2 \in M_1 \setminus \{1\}$, consider the constant family $P_{\widehat{A}} = z + \frac{1}{z} + \widehat{A},\ A \in \mathbb{C}$, and let f_λ be an analytic family of parabolic-like maps parametrized by Λ, $\Lambda \approx \mathbb{D}$. Then the set $\{(\lambda, \widehat{A}) \mid f_\lambda$ is hybrid equivalent to $P_{\widehat{A}}\} = \chi^{-1}(\widehat{B}) \times \mathbb{C}$ is an analytic subset of $M_f \times \mathbb{C}$ by the previous Proposition, and then $\chi^{-1}(\widehat{B})$ is an analytic subset of M_f. □

3.5 The map $\chi : \Lambda \to \mathbb{C}$ is a ramified covering from the connectedness locus M_f to $M_1 \setminus \{1\}$

The aim of this thesis is to prove that the map $\chi : \Lambda \to \mathbb{C}$, if not constant, restricts to a branched covering from the connectedness locus M_f to $M_1 \setminus \{1\}$. We will assume for the rest of the chapter that the map χ is not constant, and then we set $\mathcal{B} = \chi(\Lambda)$ (see 3.2.1). For every $y \in \mathcal{B}$, $\chi^{-1}(y)$ is discrete, because for all $B \in M_1 \setminus \{1\}$, $\chi^{-1}(B)$ is an analytic subset of M_f. In this section we will prove:

1. for every $\lambda \in \Lambda$, $i_\lambda(\chi) > 0$ (Prop. 3.5.4);
2. the map χ *locally* has a degree and the lifting property and if the local degree is 1 it is a local homeomorphism (Prop. 3.5.5);
3. the critical points form a discrete set (Cor. 3.5.1)
4. for all closed and connected subset M of $\mathcal{B}$, if $P = \chi^{-1}(M)$ is compact, then $\chi_{|P}$ is proper of degree equal to the sum of local degrees (Prop. 3.5.6);
5. if $(f_\lambda)_{\lambda\in\Lambda}$ is nice family of parabolic-like maps (see Def 3.5.7), the map $\chi : M_f \to M_1 \setminus \{1\}$ is a *degree* $\mathcal{D} > 0$ *branched covering* (Thm. 3.5.9, where $\mathcal{D} > 0$ is given by 3.5.10).

Notation. *Without specifications, we consider a neighborhood open.*

Let us start by reminding the notion of degree and of ramified covering.

Degree

Let X, Y be oriented topological surfaces and $\phi : X \to Y$ be a continuous map. If ϕ is *proper*, and X, Y are *connected* then ϕ has a degree. Indeed, since ϕ is continuous the induced map $\phi_* : H^2(Y) \to H^2(X)$ is a homomorphism, since X, Y are surfaces $H^2_c(X) \approx \mathbb{Z}$, $H^2_c(Y) \approx \mathbb{Z}$ (see [H] pg.134), and since ϕ is proper the induced map $\phi_* : H^2_c(Y) \to H^2_c(X)$ is of the form:

$$\alpha \to d\alpha$$

for some integer d depending only on ϕ, which is called the *degree of* ϕ, degϕ.

On the other hand, if X, Y are oriented topological surfaces (or *open* subsets of $\mathbb{C}$), ϕ is proper, X, Y are connected and for all $y \in Y$, $\phi^{-1}(y)$ is discrete, then $\phi^{-1}(y)$ is finite (since ϕ is proper) and the following formula holds (see [H] pg. 136):

$$\deg \phi = \sum_{x \in \phi^{-1}(y)} i_x(\phi),$$

where $i_x(\phi)$ is the *local degree of* ϕ *at* x, which is defined as follows: choose neighborhoods U, V of x, y respectively, homeomorphic to $\mathbb{D}$ and such that $\phi(U) \subset V$ and $\{x\} = U \cap \phi^{-1}(y)$. If γ is a loop in $U \setminus \{x\}$ with winding number 1, then $i_x(\phi)$ is the winding number of $\phi(\gamma)$ around y.

Remark 3.5.1. *Note that, if X and Y are closed sets, ϕ is proper, X, Y are connected and for all $y \in Y$, $\phi^{-1}(y)$ is discrete and finite, the equality* $\deg \phi = \sum_{x\in\phi^{-1}(y)} i_x(\phi)$ *does not hold in general. As a counterexample, set $X = \overline{D}(a, r) \subset \mathbb{C}$, where $a \neq 0$, $|a| < r$, and $\phi(z) = z^2$. Then ϕ is proper because X is compact, $Y = \phi(X)$ is compact because ϕ is continuous, but* $\deg \phi \neq \sum_{x\in\phi^{-1}(y)} i_x(\phi)$. *On the other hand, let $x_0 \in \mathring{D}(a, r)$, and set $y_0 = \phi(x_0)$. Since $x_0 \in \mathring{D}(a, r)$, $x_0 \cap \partial D(a, r) = \emptyset$, hence $y_0 \cap \phi(\partial D(a, r)) = \emptyset$. Therefore, there exists a neighborhood $V \approx \mathbb{D}$ of y_0 in Y such that $V \cap \phi(\partial D(a, r)) = \emptyset$. If U is the connected component of $\phi^{-1}(V)$ containing y_0, since $\phi^{-1}(V) \cap \partial D(a, r) = \emptyset$, $U \cap \partial D(a, r) = \emptyset$. The set U contains y_0, then $U \subset \overline{D}(a, r)$, therefore ϕ restricts to a proper map $\phi_{|U} : U \to V$ such that* $\deg \phi_{|U} = \sum_{x\in\phi^{-1}(y)\cap U} i_x(\phi)$. *Note that* $\deg \phi_{|U}$ *can be one or two. If* $\{x_0\} = U \cap \phi^{-1}(y_0)$, $\deg \phi_{|U} = 2$ *only if x_0 is a critical point.*

Ramified covering

Definition 3.5.1. Suppose X, Y are topological spaces. A map $p : X \to Y$ is a *covering map* if the following holds.

Every $y \in Y$ has an open neighborhood V such that its preimage $p^{-1}(V)$ can be represented as

$$p^{-1}(V) = \bigcup_{j\in J} U_j,$$

where the U_j, $j \in J$ are disjoint open subsets of X, and all mappings $p|_{U_j} : U_j \to V$ are homeomorphisms. In particular p is a local homeomorphism.

Definition 3.5.2. Suppose X, Y are topological spaces. A map $p : X \to Y$ is a *branched covering map* if every $y \in Y$ has a punctured neighborhood V such that $p : p^{-1}(V) \to V$ is a covering map.

Definition 3.5.3. Suppose X, Y are topological spaces, and $p : X \to Y$ is a branched covering map. A point $x \in X$ is called a *branch point* if there is no neighborhood U of x such that $p|_U$ is injective.

Proposition 3.5.4. *Let $\boldsymbol{f} = (f_\lambda)_{\lambda\in\Lambda}$ be an analytic family of parabolic-like mappings of degree 2. Then for every $\lambda \in \Lambda$, $i_\lambda(\chi) > 0$.*

Proof. The proof follows the proof of topological holomorphy of χ over M in [DH]. We can distinguish 3 cases:

1. $\lambda \in R,\ \chi(\lambda) = B \in \mathring{M}_1$ or $B \in \mathcal{B} \setminus M_1$. Since the map $\chi : \Lambda \to \mathbb{C}$ is analytic on $\mathring{M}_f$, and quasiregular on $\Lambda \setminus M_f$, $i_\lambda(\chi) > 0$.

2. $\lambda \in \mathring{M}_f,\ B \in \partial M_1$. Since χ is holomorphic on $\mathring{M}_f$, χ is open or it is constant. If χ is open there exists a neighborhood Λ' of λ in $\mathring{M}_f$, such that $\chi(\Lambda') \subset M_1$. Since $B = \chi(\lambda) \in \partial M_1$, this is impossible.

3. $\lambda \in F,\ B \in \partial M_1$. Let $\mathbb{D}$ be a disc in Λ containing λ and no other point of $\chi^{-1}(B)$. Set $\gamma = \partial\mathbb{D}$. Since $\lambda \in F$, there exists in $\mathring{\mathbb{D}}$ a λ' such that $f_{\lambda'}$ has an attracting periodic point and $B' = \chi(\lambda')$ is in the same connected component of B. Hence $i_\lambda(\chi) = \sum_{x\in\phi^{-1}(B')\cap\mathbb{D}} i_x(\chi) > 0$, because every term in the sum is positive, since χ is holomorphic at λ', and there exists at least one term in the sum, which is λ'.

$\square$

Proposition 3.5.5. *Let $\boldsymbol{f} = (f_\lambda)_{\lambda\in\Lambda}$ be an analytic family of parabolic-like mappings of degree 2, let $\lambda \in \Lambda$ and $B = \chi(\lambda)$. Then the following statements hold:*

1. *there exist open connected neighborhoods U of λ and V of $B = \chi(\lambda)$, with compact closure in Λ and $\mathcal{B}$ respectively, such that χ restricts to a proper surjective map $\chi_{|U} : U \to V$ of degree $d = i_\lambda(\chi)$;*

2. *we can write $\chi_{|U}$ as $\pi\circ\tilde{f}$, where $\pi : \tilde{V} \to V$ ($\tilde{V} \approx \mathbb{D}$) is a d-fold branched covering of V ramified above B (i.e. a branched covering with branched point $\tilde{B}$ such that $\pi(\tilde{B}) = B$), and $\tilde{f} : U \to \tilde{V}$ is a homeomorphism. In particular, if $d = 1$ the map χ restricts to a homeomorphism $\chi : U \to V$.*

Proof. 1. The proof follows the one in the context of topological holomorphy of [DH]. Since $\mathbb{C}$ is a metric space, for all $\lambda \in \Lambda$ there exists a compact neighborhood of λ in Λ. Let C be a compact neighborhood of λ in Λ such that $\{\lambda\} = C \cap \chi^{-1}(B)$. Since C is compact, $\chi : C \to K = \chi(C)$ is proper, and since χ is continuous, K is compact. The set C is a neighborhood of λ, then $\lambda \notin \partial C$, and thus $B \cap \chi(\partial C) = \emptyset$. Since the local degree of χ is positive at every parameter in Λ we can assume, taking C small if necessary, that $Ind_B(\chi(\partial C)) = i_\lambda(\chi) > 0$. Hence $\mathbb{C} \setminus \chi(\partial C)$ has a bounded connected component homeomorphic to a disc containing B, and there exists V open neighborhood of B homeomorphic to a disc such that $V \cap \chi(\partial C) = \emptyset$. Let U be the connected component of $\chi^{-1}(V)$ containing λ. Then $\chi^{-1}(V) \cap \partial C = \emptyset$, hence $U \cap \partial C = \emptyset$. Since $\{\lambda\} = C \cap \chi^{-1}(B)$ and $U \subset C$, $\chi_{|U} : U \to V$ is a proper map of degree $d = i_\lambda(\chi)$.

2. By the lifting criterion, (see [H] prop 1.33 pag. 60), if $p : (X, x_0) \to (Y, y_0)$ is a map with X path connected and locally path connected, and $\pi : (\tilde{Y}, \tilde{y_0}) \to (Y, y_0)$ is a covering space, then a lift $\tilde{p} : (X, x_0) \to (\tilde{Y}, \tilde{y_0})$ exists if and only if $p_*(\pi_1(X, x_0)) \subseteq \pi_*(\pi_1(\tilde{Y}, \tilde{y_0}))$. Then we need $\chi_*((\pi_1(U \setminus \{\lambda\})) \subseteq \pi_*(\pi_1(\tilde{V} \setminus \{\tilde{B}\}))$. Note that, by (1), χ induces a proper surjective map between U and V of degree $d = i_\lambda(\chi)$. Hence, since $\pi_1(U \setminus \{\lambda\}) = \mathbb{Z} = \pi_1(V \setminus \{B\})$, the mapping $\chi_* : \pi_1(U \setminus \{\lambda\}) \to \pi_1(V \setminus \{B\})$ is multiplication by the integer $d = i_\lambda(\chi)$. Similarly, $\pi_1(\tilde{V} \setminus \{\tilde{B}\}) = \mathbb{Z}$ and the map $\pi_* : \pi_1(\tilde{V} \setminus \{\tilde{B}\}) \to \pi_1(V \setminus \{B\})$ is multiplication by the integer d, since $\pi : \tilde{V} \to V$ is the projection of the d-folder cover of V. Therefore $\chi_*(\pi_1(U \setminus \{\lambda\})) = d\mathbb{Z} = \pi_*(\pi_1(\tilde{V} \setminus \{\tilde{B}\}))$, and finally there exists a lift of χ to π. By openess $\tilde{f}$ is a homeomorphism, and then if $d = 1$ the map χ restricts to a homeomorphism $\chi : U \to V$.

□

Corollary 3.5.1. *In the notation of the above Proposition, the critical points of χ, i.e. the points of Λ where $i_\lambda(\chi) > 1$, form a closed discrete subset of Λ.*

Proof. Suppose $\lambda \in U \cap \Lambda$ is a critical point. If $q \in U \cap \Lambda$ and $q \neq \lambda$, then $i_q(\chi) = i_q(\tilde{f}) = 1$ (since $\tilde{f}$ is a homeomorphism and π is ramified only above B). Indeed by Prop. 3.5.5, $\chi(q) = \pi \circ \tilde{f} = n \neq B$, and since π is a covering branched at B, there exists a neighborhood $U(n)$ in V such that $\pi^{-1}(U(n)) = \bigcup_{j \in J} U_j$, and all mappings $\pi|_{U_j} : U_j \to U(n)$ are homeomorphisms. In particular $i_{\tilde{f}(q)}(\pi) = 1$ and thus $i_q(\chi) = i_q(\tilde{f}) i_{\tilde{f}(q)}(\pi) = i_q(\tilde{f})$. Trivially, this set is closed since its complement (the set of points of Λ where $i_\lambda(\chi) = 1$) is an open set (indeed if $\lambda' \in \Lambda$ has $i_{\lambda'}(\chi) = 1$, then here exists a neighborhood $U(\lambda')$ of λ' such that $\forall z \in U(\lambda'),\ i_z(\chi) = 1)$. Hence λ is the only critical point in $U \cap \Lambda$. $\square$

Proposition 3.5.6. *Let M be a closed and connected subset of $\mathcal{B}$, and $P = \chi^{-1}(M)$. If P is compact, then there exist neighborhoods $\hat{V}$ of M in $\mathcal{B}$ and $\hat{U}$ of P in Λ such that $\chi : \hat{U} \to \hat{V}$ is a proper map of degree d, where, for any $m \in M$, $d = \sum_{p \in \chi^{-1}(m)} i_p(\chi)$.*

Proof. The proof follows the one in the context of topological holomorphy of [DH]. Since P is compact, $P \subset \Lambda \subset \mathbb{C}$, and $P \cap \partial\Lambda = \emptyset$, the distance $r = dist(P, \partial\Lambda)$ is positive. Let N be a closed neighborhood of P in Λ with $dist(P, \partial N) = r/2 = dist(N, \partial\Lambda)$. Hence $P \subset N \subset \Lambda$, and $\chi : N \to \chi(N)$ is proper. Since $P = \chi^{-1}(M)$, and $\partial P \cap \partial N = \emptyset$, $\partial M \cap \chi(\partial N) = \emptyset$. Call $\hat{V}$ the connected component of $\mathcal{B} \setminus \chi(\partial N)$ which contains M, and set $\hat{U} = \chi^{-1}(\hat{V}) \cap N$. Then $\chi^{-1}(\hat{V}) \cap \partial N = \emptyset$, hencethe map $\chi_{|\hat{U}} : \hat{U} \to \hat{V}$ is proper. The map χ is continuous, hence, since $\hat{V}$ is connected, $\hat{U}$ is the union of connected components. Let us set $\hat{U} = \bigcup_j \hat{U}_j$. The restriction $\chi : \hat{U}_j \to \hat{V}$ is then a proper map between connected sets, thus it has a degree, which we call d_j. Note that, for all j, $d_j > 0$. Therefore $\chi : \hat{U} \to \hat{V}$ has a degree:

$$d = \deg \chi_{|\hat{U}} = \sum_j d_j$$

Moreover, since $\hat{U},\ \hat{V}$ open, $\chi : \hat{U} \to \hat{V}$ proper and for every $v \in \hat{V}$, $\chi^{-1}(v)$ is discrete and finite,

$$d = \deg \chi_{|\hat{U}} = \sum_{u \in \chi^{-1}(v) \cap \hat{U}} i_u(\chi).$$

Hence for all $m \in M$, $d = \deg \chi_{|\hat{U}} = \deg \chi_{|P} = \sum_{p \in \chi^{-1}(m) \cap P} i_p(\chi)$. $\square$

3.5.1 Nice families of parabolic-like maps

As we saw in 3.2.1, the range $\mathcal{B}$ of the map χ is not the whole of $\mathbb{C}$, but a proper subset of $\mathbb{C}$, because there is no $\lambda \in \Lambda$ such that f_λ is hybrid equivalent to $P_0 = z + 1/z$. Hence $M_1 \not\subseteq \mathcal{B}$, since the root $B = 1$ does not belong to $\mathcal{B}$. However, we could hope that, for all $B \in \mathcal{B}$, either

1. $B \notin M_1$ as $B \to \partial\mathcal{B}$, or
2. $B \to 1$ as $B \to \partial\mathcal{B}$.

Indeed this is the case under appropriate conditions (e.g. the following one).

Definition 3.5.7. Let $\mathbf{f} = (f_\lambda : U'_\lambda \to U_\lambda)_{\lambda\in\Lambda}$ be an analytic family of parabolic-like maps of degree 2, such that, for $\lambda \to \partial\Lambda$:

1. $\lambda \notin M_f$ or
2. $\chi(\lambda) \to 1$.

Then we call $\mathbf{f}$ a *nice family of parabolic-like mappings.*

Proposition 3.5.8. *Let $\boldsymbol{f}$ be a nice family of parabolic-like mappings. Then, for every $U(1)$ neighborhood of 1 in $\mathbb{C}$, setting $K = M_1 \setminus U(1)$, the set $C = \chi^{-1}(K)$ is compact in Λ.*

Proof. Assume C is not compact in Λ. Then there exists a sequence $(\lambda_n) \in C$ such that $\lambda_n \to \partial\Lambda$ as $n \to \infty$. On the other hand, for all n, $\chi(\lambda_n) \in K$. Let $\chi(\lambda_{n_k})$ be a subsequence converging to some parameter B. Since K is compact, the limit point B belongs to $K \subset M_1 \setminus \{1\}$. This is a contradiction, because $\mathbf{f}$ is a nice family of parabolic-like mappings. Therefore C is compact in Λ. □

If $\mathbf{f}$ is a nice family of parabolic-like mappings, $U(1)$ a neighborhood of the root of M_1, $K = M_1 \setminus U(1)$, and $C = \chi^{-1}(K)$, by Prop. 3.5.6 there exist neighborhoods $\hat{U}$ of C in Λ and $\hat{V}$ of K in $\mathcal{B}$ such that the restriction $\chi : \hat{U} \to \hat{V}$ is a proper map of degree $\mathcal{D}$.

Theorem 3.5.9. *Given a nice family of parabolic-like maps $f_{\lambda,\lambda\in\Lambda\approx\mathbb{D}}$, the map $\chi : M_f \to M_1 \setminus \{root\}$ is a degree $\mathcal{D} > 0$ branched covering. More precisely, given K compact and connected with $M_1 \setminus U(1) \subset K \subset \mathcal{B}$ and $0 \in K$, there exists a $\hat{V}$ neighborhood of K in $\mathcal{B}$ such that the map $\chi : \hat{U} = \chi^{-1}(\hat{V}) \to \hat{V}$ is a degree $\mathcal{D} > 0$ branched covering.*

Proof. We want to prove that, for all $y \in \hat{V}$, there exists a punctured neighborhood $V^*(y)$ of y in $\hat{V}$ such that $\chi : \chi^{-1}(V^*(y)) \to V^*(y)$ is a covering map, i.e. for all $z \in (V^*(y))$ there exists a neighborhood $V(z)$ of z in $\hat{V}$ such that $\chi^{-1}(V(z)) = \bigcup_{j\in J} U_j$, where U_j, $j \in J$ are disjoint subsets of $\hat{U}$, and all mappings $\chi|_{U_j} : U_j \to V(z)$ are homeomorphisms.

By Prop. 3.5.6 the map $\chi : \hat{U} \to \hat{V}$ is a proper map of degree $\mathcal{D}$. Let $y \in \hat{V}$. By Corollary 3.5.1, the set of $x \in \Lambda$ with $i_x(\chi) > 1$ is a closed discrete set, hence there exists a punctured neighborhood of $V^*(y)$ of y in $\hat{V}$ such that, for all $x \in \chi^{-1}(V^*(y))$, $i_x(\chi) = 1$. Call $U_1^*, ..., U_{\mathcal{D}}^*$ the preimages of $V^*(y)$. Let $z \in V^*(y)$, and let $z_1, ..., z_{\mathcal{D}}$ be the preimages of z in $U_1^*, ..., U_{\mathcal{D}}^*$ respectively. Hence, by Prop. 3.5.5(1), for all $i \leq \mathcal{D}$ there exists neighborhoods $U(z_i) \subset \hat{U}$ and $V_i(z) \subset \hat{V}$ of z_i and z respectively such that the map χ induces a homeomorphism $\chi : U(z_i) \to V_i(z)$. Define $V(z) = \bigcap_i V_i(z)$, then $\chi^{-1}(V(z)) = \bigcup_{0<i\leq\mathcal{D}} U_i$, where the U_i are disjoint subsets of $\hat{U}$, and all mappings $\chi|_{U_i} : U_i \to V(z)$ are homeomorphisms.

□

Following the polynomial-like setting, we call $\mathcal{D}$ the *parametric degree* of the family $\mathbf{f}$. Note that, if $\mathcal{D} = 1$, then χ restricts to a homeomorphism $M_f \to M_1 \setminus \{root\}$. Next Proposition tells us how to compute the *parametric degree* $\mathcal{D}$ of the family $\mathbf{f}$.

Proposition 3.5.10. *Let $f_{\lambda,\lambda\in\Lambda\approx\mathbb{D}}$ be a nice family of parabolic-like maps, K be a compact and connected set with $M_1 \setminus U(1) \subset K \subset \mathcal{B}$ and $0 \in K$, and set $C = \chi^{-1}(K)$. Let $\hat{V}$ be a neighborhood of K in $\mathcal{B}$ and set $\hat{U} = \chi^{-1}(\hat{V})$. The degree $\mathcal{D}$ of the branched covering $\chi : \hat{U} \to \hat{V}$ is equal to the number of times $f_\lambda(c_\lambda) - c_\lambda$ turns around 0 as λ describes ∂C.*

Let us remind that for every A the map $P_A = z + 1/z + A$ has two critical points: $z = 1$ and $z = -1$. After a change of coordenates we can assume $z = 1$ is the first critical point attracted by ∞. Hence for all $A \in \mathbb{C}$, $z = -1$ is the critical point in the parabolic-like restriction of P_A (see the proof of 2.5.1).

Proof. The proof follows the one in [DH].

Let c_λ be the critical point of f_λ. Choose λ_0 such that $f_{\lambda_0}(c_{\lambda_0}) = c_{\lambda_0}$. Let $[P_{A_0}]$ be the member of the family $Per_1(1)$ hybrid equivalent to f_{λ_0}. Therefore $P_{\pm A_0}(-1) = -1$. An easy computation shows that $\chi(\lambda_0) = B_0 = 0$. This means that the multiplicity of λ_0 as zero of $\lambda \to \chi(\lambda)$ is the multiplicity of λ_0 as zero of the map $\lambda \to f_\lambda(c_\lambda) - c_\lambda$. Hence $\mathcal{D} = \sum_{\lambda\in\chi^{-1}(0)} i_\lambda(\chi)$ is the number of zeroes of the map $\lambda \to f_\lambda(c_\lambda) - c_\lambda$ counted with multiplicity. □

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